MODULARITY OF TRIANGULINE GALOIS REPRESENTATIONS

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Abstract. We use the theory of trianguline (φ, Γ)-modules over pseudorigid spaces to prove a modularity lifting theorem for certain Galois representations which are trianguline at p, including those with characteristic p coefficients. The use of pseudorigid spaces lets us construct integral models of the trianguline varieties of [BHS17], [Che13] after bounding the slope, and we carry out a Taylor–Wiles patching argument for families of overconvergent modular forms. This permits us to construct a patched quaternionic eigenvariety and deduce our modularity results.

1. Introduction

The Fontaine–Mazur conjecture predicts that representations of Galois groups of number fields which are sufficiently nice should come from geometry. In practice, the way one proves this is by proving so-called automorphy lifting theorems, relating the Galois representations of interest to Galois representations already known to have the desired properties.

In this context, if $\rho : \mathrm{Gal}_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is the representation, "sufficiently nice" includes a condition on the local Galois group at p called being *geometric*. In the present paper, motivated by a question of Andreatta–Iovita–Pilloni [AIP18], we consider a characteristic p analogue of this conjecture. There is no definition of "geometric" for a Galois representation with positive characteristic coefficients, but we replace it with the condition *trianguline*:

Theorem. *Assume $p \geq 5$, and let L be a finite extension of $\mathbf{F}_p(\!(u)\!)$. Let $\rho : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathscr{O}_L)$ be an odd continuous Galois representation unramified away from p such that the (φ, Γ)-module $D_{\mathrm{rig}}(\rho|_{\mathrm{Gal}_{\mathbf{Q}_p}})$ is trianguline with regular parameters. Assume moreover that the reduction $\overline{\rho}$ is modular and satisfies certain additional technical hypotheses. Then ρ is the twist of the Galois representation corresponding to a point on the extended eigencurve $\mathscr{X}_{\mathrm{GL}_2}$.*

The eigencurve $\mathscr{X}_{\mathrm{GL}_2}^{\mathrm{rig}}$ was originally constructed by Coleman–Mazur, and it is a rigid analytic space whose points correspond to *overconvergent modular forms*. Points corresponding to classical eigenforms (of varying weight and level) are dense, so we can think of it as a moduli space of p-adic modular forms. Each point of the eigencurve has a Galois representation attached, but Kisin [Kis03] showed that the Galois representations at non-classical points are not geometric at p. Instead, they are trianguline (though he did not use this terminology; it was introduced subsequently by Colmez). A converse was proved by Emerton [Eme11, Theorem 1.2.4] when the coefficients are p-adic.

Given a p-adic Galois representation ρ, there is an associated object $D_{\mathrm{rig}}(\rho)$ called a (φ, Γ)-module; at the expense of making the coefficients more complicated, the Galois representation can be captured as the action of a semi-linear operator φ together with the action of a 1-dimensional p-adic Lie group Γ. Then even if ρ is irreducible, it is possible for $D_{\mathrm{rig}}(\rho)$ to be reducible. Kisin showed that this happens in small neighborhoods of classical points on the eigencurve; if ρ_x is the Galois representation attached to a point x, there is an exact sequence

$$0 \to D_1 \to D_{\mathrm{rig}}(\rho_x) \to D_2 \to 0$$

where D_1 and D_2 are rank-1 (φ, Γ)-modules. There is a basis element $\mathbf{e}_1$ of D_1 such that φ acts on $\mathbf{e}_1$ by the U_p-eigenvalue at x and Γ acts on $\mathbf{e}_1$ trivially. This construction was extended over (a normalization of) the eigencurve in separate work of [KPX14] and [Liu15].

The eigencurve is equipped with a map $\mathrm{wt} : \mathscr{X}_{\mathrm{GL}_2}^{\mathrm{rig}} \to \mathscr{W}^{\mathrm{rig}}$ to *weight space*, which we may view as the disjoint union of $p-1$ rigid analytic open unit disks. The existence of Galois representations attached to eigenforms means it is also equipped with a morphism $\mathscr{X}_{\mathrm{GL}_2}^{\mathrm{rig}} \to \mathbf{G}_m^{\mathrm{rig}} \times \coprod_{\overline{\rho}} R_{\overline{\rho}}$, where the $R_{\overline{\rho}}$ are Galois deformation rings (more precisely, deformation rings of pseudocharacters), and $\mathbf{G}_m^{\mathrm{rig}}$ corresponds to the eigenvalue of the Hecke operator U_p. The triangulation results of [Kis03], [KPX14], and [Liu15] mean that we can combine these two maps to get a morphism

$$\mathscr{X}_{\mathrm{GL}_2} \to \coprod_{\overline{\rho}} X_{\mathrm{tri},\overline{\rho}}^{\psi,\kappa,\mathrm{rig}}$$

to a moduli space of trianguline Galois representations (here the decorations ψ and κ simply mean we are fixing the determinant and the parameters of the triangulation). The result of [Eme11] then shows that this morphism surjects onto certain components.

More recently, the construction of the eigencurve has been extended to mixed characteristic by Andreatta–Pilloni–Iovita [AIP18], [AIP16] and Johansson–Newton [JN16], using Huber's theory of adic spaces instead of Tate's theory of rigid analytic spaces. These authors construct *pseudorigid spaces* containing characteristic 0 eigenvarieties as open subspaces, with non-empty characteristic p loci.

In previous work, we generalized the construction of (φ, Γ)-module to families of Galois representations with pseudorigid coefficients [Bel20] and showed that the triangulation of the eigencurve extends to the boundary characteristic p points [Bel21]. This yields an analogous morphism $\mathscr{X}_{\mathrm{GL}_2} \to \coprod X^{\psi,\kappa}_{\mathrm{tri},\overline{\rho}}$ of pseudorigid spaces. In the present paper, we use that machinery to prove a modularity result for Galois representations trianguline at p, characterizing the image in many components.

The proof rests on the Taylor–Wiles patching method, as reformulated in [Sch18]. This is the source of the aforementioned technical hypotheses on $\overline{\rho}$ (which amount to assumptions about the image of $\overline{\rho}$ being sufficiently big). However, there are a number of technical complications. For example, to carry out some preliminary reductions, we first prove a version of the Jacquet–Langlands correspondence on eigenvarieties extending the construction of [Bir19], and we characterize the image of the cyclic base change morphism $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q}} \to \mathscr{X}_{\mathrm{GL}_2/F}$ of [JN19a]. The latter uses the construction of an auxiliary "$\mathrm{Gal}(F/\mathbf{Q})$-fixed" eigenvariety, which may be of independent interest. This permits us to transfer the problem to overconvergent quaternionic modular forms over a cyclic totally real extension of $\mathbf{Q}$.

The modules of quaternionic automorphic forms we patch are those constructed in [JN16]. We construct trianguline deformation rings which act on them, and we patch by introducing ramification at additional primes. But the construction of trianguline deformation rings is delicate, because in general triangulations of (φ, Γ)-modules do not interact well with integral structures on the corresponding Galois representation. Thus, we crucially use the pseudorigid theory of triangulations (and not just the rigid analytic theory) to ensure that we can construct an integral quotient of a Galois deformation ring whose analytic points are trianguline, with Frobenius eigenvalues bounded by a fixed slope.

This leads to a further difficulty, which is that it is difficult to study the components of the trianguline deformation ring directly. Instead, we patch families of overconvergent automorphic forms, which lets us compare the Galois representation we are interested in with "nearby" representations which are known to be automorphic. Along the way, we construct local pieces of a patched quaternionic eigenvariety $\mathscr{X}^{\infty}_{\underline{D}^\times}$, together with a morphism to a trianguline variety and a patched module of overconvergent modular forms. We note that it is only possible to patch families of overconvergent automorphic forms because we constructed an integral model of the trianguline variety; we know almost nothing about its structure away from nice points in the analytic locus, but understanding it better would be very interesting.

We have not attempted to work in maximum generality. In particular, it should be possible to relax the ramification condition and prove an overconvergent modularity lifting theorem for certain totally real fields. However, this would require constructing and studying a cyclic base change morphism for more general extensions of number fields. We expect that it is possible to construct these morphisms for the middle-degree eigenvariety over a totally real field, which would lead to stronger trianguline modularity theorems in characteristic 0. But we were forced to assume the degree of the cyclic extension was prime to p to characterize the image of a base change morphism in positive characteristic, so additional work would be required to strengthen our results in positive characteristic.

We further remark that our "big image" condition on the residual Galois representation is stronger than the standard one. This is to ensure that we have access to the necessary cohomological vanishing theorems, to permit us to work with middle-degree eigenvarieties.

The work of Breuil–Hellmann–Schraen [BHS17] constructs a similar patched eigenvariety for unitary groups, using completed cohomology rather than overconvergent cohomology. It would be extremely interesting to relate these two constructions.

We now describe the structure of this paper. We begin by recalling the theory of trianguline (φ, Γ)-modules and their deformations; this permits us to construct and study pseudorigid trianguline varieties (generalizing those of [Che13] and [BHS17]). We compute the dimension of these pseudorigid trianguline varieties with fixed determinant and weight, and we show that they have an integral model after bounding the slopes of the rank-1 constituents.

We then turn to the automorphic theory we will need. We prove that so-called *twist classical* points are very Zariski dense in the eigenvariety $\mathscr{X}_{\underline{D}^\times}$, which permits us to interpolate the Jacquet–Langlands correspondence to extended eigenvarieties and permits us to conclude that $\mathscr{X}_{\underline{D}^\times}$ is reduced (extending the results of [Bir19] and [Che05]). We also study the cyclic base change morphism $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q}} \to \mathscr{X}_{\mathrm{GL}_2/F}$ of [JN19a]; when F is totally real and $[F : \mathbf{Q}]$ is prime to p, we show that $x \in \mathscr{X}_{\mathrm{GL}_2/F}$ is in the image if and only if it is fixed by $\mathrm{Gal}(F/\mathbf{Q})$. To do this, we construct a "$\mathrm{Gal}(F/\mathbf{Q})$-fixed eigenvariety" and show that classical points are dense in it.

Finally, we turn to the patching argument. We show that our modules of integral overconvergent automorphic forms are projective, and we show that we can add certain kinds of level structure. Then using the standard Taylor–Wiles patching construction, we construct a patched module with the support we expect. This permits us to deduce the desired modularity statement, by interpolation from crystalline points in characteristic 0. This last step requires the results of [Kis09a], which in turn requires the p-adic local Langlands correspondence of [Eme11]. Thus, while our argument applies to characteristic 0 Galois representations, it does not replace the trianguline modularity result of that paper.

Notation. We fix some running notation and hypotheses. In section 2 we assume that $p \geq 3$, because we only developed the theory of (φ, Γ)-modules over pseudorigid spaces in that situation. In sections 3 and 5, we assume

$p \geq 5$; we need this hypothesis to construct eigenvarieties (and the Jacquet–Langlands and cyclic base change morphisms between them) at tame level 1, and later to apply Taylor–Wiles patching.

We normalize class field theory so that it sends uniformizers to geometric Frobenius, and we normalize Hodge–Tate weights so that the cyclotomic character has Hodge–Tate weight -1.

If X is a group isomorphic to $X_0 \times \mathbf{Z}_p^{\oplus r} \times \mathbf{Z}^{\oplus s}$, where X_0 is a finite group, we let $\widehat{X} := \underline{\mathrm{Hom}}(X, \mathbf{G}_m^{\mathrm{ad}})$ denote the functor $R \mapsto \mathrm{Hom}_{\mathrm{cts}}(X, R^\times)$.

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2. Trianguline varieties and Galois deformation rings

2.1. **Galois deformation rings.** Let $\mathbf{E}/\mathbf{Q}_p$ be a finite extension, with ring of integers $\mathscr{O}_E$, uniformizer ϖ_E, and residue field $\mathbf{F}$, and let G be a profinite group satisfying Mazur's condition Φ_p. The two cases we will be most interested in are $G = \mathrm{Gal}_K$ and $G = \mathrm{Gal}_{F,S}$, where K is a finite extension of $\mathbf{Q}_p$, and F is a number field, and S is a set of places of F.

Suppose we have a continuous homomorphism $\overline{\rho} : G \to \mathrm{GL}_d(\mathbf{F})$. Then we may construct the univeral framed deformation ring $R_{\overline{\rho}}^{\square}$, which pro-represents the functor

$$A \rightsquigarrow \{\rho : G \to \mathrm{GL}_d(A) \mid \rho \equiv \overline{\rho} \pmod{\mathfrak{m}_A}\}$$

on the category of complete local noetherian $\mathscr{O}_E$-algebras with residue field $\mathbf{F}$, of lifts of $\overline{\rho}$, that is, deformations of $\overline{\rho}$ together with a basis. If $\mathrm{End}_G(\overline{\rho}) = \mathbf{F}$ (for example, if $\overline{\rho}$ is absolutely irreducible), we additionally have the universal (unframed) deformation ring $R_{\overline{\rho}}$ parametrizing deformations of ρ.

If R is a complete local noetherian $\mathscr{O}_E$-algebra with maximal ideal $\mathfrak{m}_R$ and finite residue field, and $\psi : \mathrm{Gal}_K \to R^\times$ is a continuous character such that $\det \overline{\rho} = \psi \mod \mathfrak{m}_R$, there is a quotient $R \widehat{\otimes} R_{\overline{\rho}}^{\square} \twoheadrightarrow R_{\overline{\rho}}^{\square,\psi}$ parametrizing lifts of $\overline{\rho}$ with determinant ψ. Indeed, there is a homomorphism $R_{\det \overline{\rho}} \to R_{\overline{\rho}}^{\square}$ given by the determinant map, and the choice of ψ defines a homomorphism $R_{\det \overline{\rho}} \to R$; then $R_{\overline{\rho}}^{\square,\psi} = R \widehat{\otimes}_{R_{\det \overline{\rho}}} R_{\overline{\rho}}^{\square}$. If $\mathrm{End}_G(\overline{\rho}) = \mathbf{F}$, there is similarly a quotient $R \widehat{\otimes} R_{\overline{\rho}} \twoheadrightarrow R_{\overline{\rho}}^{\psi}$ parametrizing deformations of $\overline{\rho}$ with determinant ψ.

Now we specialize to the arithmetic situations of interest. Let $K/\mathbf{Q}_p$ be a finite extension, and assume that $\mathrm{Hom}(K, E)$ has cardinality $[K : \mathbf{Q}_p]$. Then by [BIP21, Corollary 3.37], $R_{\overline{\rho}}^{\square}$ is a complete intersection, and by [BIP21, Corollary 4.21] the irreducible components of $\operatorname{Spec} R_{\overline{\rho}}^{\square}$ are in bijection with the irreducible components of $\operatorname{Spec} R_{\det\overline{\rho}}$. More precisely, if $\mu := \mu_{p^\infty}(K)$ denotes the p-power roots of unity in $K^\times$, local class field theory identifies it with a subgroup of $\mathrm{Gal}_K^{\mathrm{ab}}$; by [BIP21, Lemma 4.1] $R_{\det\overline{\rho}}$ is a power series ring over $\mathscr{O}_E[\mu]$, so its irreducible components are in bijection with characters $\chi : \mu \to \mathscr{O}_E^\times$. There are quotients $R_{\overline{\rho}}^{\square} \twoheadrightarrow R_{\overline{\rho}}^{\square,\chi}$ parametrizing lifts of $\overline{\rho}$ whose determinant restricted to μ (via the Artin map) agrees with χ, and by [BIP21, Corollary 4.5, Corollary 4.19] the rings $R_{\overline{\rho}}^{\square,\chi}$ are normal domains and complete intersections. In particular, $R_{\overline{\rho}}^{\square}$ is reduced.

Let F be a number field and let $\Sigma_p := \{v \mid p\}$. If $\rho : \mathrm{Gal}_F \to \mathrm{GL}_d(\mathbf{F})$ is a continuous representation and v is a place of F, we let ρ_v denote $\rho|_{\mathrm{Gal}_{F_v}}$. Suppose that $\overline{\rho}$ is absolutely irreducible, and let S be a finite set of places of F containing Σ_p and the infinite places such that $\overline{\rho}$ is unramified outside S. Then we let $R_{\overline{\rho},S}$ denote the universal deformation ring parametrizing deformations unramified outside of S, and we let $R_{\overline{\rho},S}^{\square}$ denote the universal deformation ring whose A-points are deformations ρ_A of $\overline{\rho}$ unramified outside of S, together with bases for $\rho_A|_{\mathrm{Gal}_{F_v}}$ for each $v \in \Sigma_p$. We also let $R_{\overline{\rho},\mathrm{loc}}^{\square} := \otimes_{v\in\Sigma_p} R_{\overline{\rho}_v}^{\square}$.

If $\psi : \mathrm{Gal}_F \to R^\times$ is a continuous character as above, we let

$$R_{\overline{\rho},S}^{\psi} := R \mathop{\widehat{\otimes}}_{R_{\det\overline{\rho},S}} R_{\overline{\rho},S}$$

$$R_{\overline{\rho},S}^{\square,\psi} := R \mathop{\widehat{\otimes}}_{R_{\det\overline{\rho},S}} R_{\overline{\rho},S}^{\square}$$

$$R_{\overline{\rho},\mathrm{loc}}^{\square,\psi} := R \mathop{\widehat{\otimes}}_{R_{\det\overline{\rho},\mathrm{loc}}} R_{\overline{\rho},\mathrm{loc}}^{\square}$$

For any place $v \in \Sigma_p$, restriction from $\mathrm{Gal}_{F,S}$ to Gal_{F_v} defines a homomorphism $R_{\overline{\rho}_v}^{\square} \to R_{\overline{\rho},S}^{\square}$, and so we obtain homomorphisms

$$R_{\overline{\rho},\mathrm{loc}}^{\square} \to R_{\overline{\rho},S}^{\square}$$

and

$$R_{\overline{\rho},\mathrm{loc}}^{\square,\psi} \to R_{\overline{\rho},S}^{\square,\psi}$$

We can relate our local and global deformation rings more precisely:

Lemma 2.1.1. *Suppose that $p \nmid d$. Let h^1 denote the dimension (as an $\mathbf{F}$-vector space) of*

$$\ker\left(H^1(\mathrm{Gal}_{F,S}, \mathrm{ad}^0(\overline{\rho})) \to \prod_{v\in\Sigma_p} H^1(\mathrm{Gal}_{F_v}, \mathrm{ad}^0(\overline{\rho}_v))\right)$$

let $\delta_F := \dim_{\mathbf{F}} H^0(\mathrm{Gal}_{F,S}, \mathrm{ad}\,\overline{\rho})$, and for $v \in \Sigma_p$ let $\delta_v := \dim_{\mathbf{F}} H^0(\mathrm{Gal}_{F_v}, \mathrm{ad}\,\overline{\rho}_v)$. Then $R_{\overline{\rho},S}^{\square,\psi}$ can be topologically generated over $R_{\overline{\rho},\mathrm{loc}}^{\square,\psi}$ by $g := h^1 + \sum_{v\in\Sigma_p} \delta_v - \delta_F$ elements.

Proof. Let $\mathfrak{m}_{\mathrm{loc}}$ denote the maximal ideal of $R^{\square,\psi}_{\overline{\rho},\mathrm{loc}}$ and let $\mathfrak{m}_S$ denote the maximal ideal of $R^{\square,\psi}_{\overline{\rho},S}$. We need to compute the relative tangent space $\left(\mathfrak{m}_S/(\mathfrak{m}_S^2,\mathfrak{m}_{\mathrm{loc}})\right)^*$ of $R^{\square,\psi}_{\overline{\rho},S}/\mathfrak{m}_{\mathrm{loc}}$. But the maximal ideal of R is contained in $\mathfrak{m}_{\mathrm{loc}}$, so we may assume that ψ is constant, and the result follows from [Kis09b, Lemma 3.2.2]. $\square$

2.2. **Deformations of trianguline (φ,Γ)-modules.** Trianguline (φ,Γ)-modules are those which are extensions of (φ,Γ)-modules of character type. More precisely,

Definition 2.2.1. Let X be a pseudorigid space over $\mathscr{O}_E$ for some finite extension $E/\mathbf{Q}_p$, let $K/\mathbf{Q}_p$ be a finite extension, and let $\underline{\delta}=(\delta_1,\dots,\delta_d):(K^\times)^d\to\Gamma(X,\mathscr{O}_X^\times)$ be a d-tuple of continuous characters. A (φ,Γ_K)-module D is *triangulіne with parameter* $\underline{\delta}$ if (possibly after enlarging E) there is an increasing filtration $\mathrm{Fil}^\bullet D$ by (φ,Γ_K)-modules and a set of line bundles $\mathscr{L}_1,\dots,\mathscr{L}_d$ such that $\mathrm{gr}^i D\cong\Lambda_{X,\mathrm{rig},K}(\delta_i)\otimes\mathscr{L}_i$ for all i.

If $X=\mathrm{Spa}\,R$ where R is a field, we say that D is *strictly triangulіne with parameter* $\underline{\delta}$ if for each i, $\mathrm{Fil}^{i+1}D$ is the unique sub-(φ,Γ_K)-module of D containing $\mathrm{Fil}^i D$ such that $\mathrm{gr}^{i+1}D\cong\Lambda_{R,\mathrm{rig},K}(\delta_{i+1})$.

As in the characteristic 0 situation treated in [BC09, §2.3], we may define and study deformations of triangulіne (φ,Γ)-modules:

Definition 2.2.2. Let R be a finite extension of $\mathbf{F}_p(\!(u)\!)$ and let D be a fixed (φ,Γ_K)-module of rank d over $\Lambda_{R,\mathrm{rig},K}$ equipped with a triangulation $\mathrm{Fil}^\bullet D$ with parameter $\underline{\delta}$. Let $\mathcal{C}_R$ denote the category of artin local $\mathbf{Z}_p$-algebras R' equipped with an isomorphism $R'/\mathfrak{m}_{R'}\xrightarrow{\sim}R$. The triangulіne deformation functor $\mathrm{Def}_{D,\mathrm{Fil}^\bullet}:\mathcal{C}_R\to\underline{\mathrm{Set}}$ is defined to be the set of isomorphism classes

$$\mathrm{Def}_{D,\mathrm{Fil}^\bullet}(R'):=\{(D_{R'},\mathrm{Fil}^\bullet D_{R'},\iota)\}/\sim$$

where $D_{R'}$ is a (φ,Γ_K)-module over $\Lambda_{R',\mathrm{rig},K}$, $\mathrm{Fil}^\bullet D_{R'}$ is a triangulation, and $\iota:R\otimes_{R'}D_{R'}\xrightarrow{\sim}D$ is an isomorphism which also defines isomorphisms $R\otimes_{R'}\mathrm{Fil}^i D_{R'}\xrightarrow{\sim}\mathrm{Fil}^i D$.

One of the consequences of the proof of [Bel20, Proposition 5.1] is that when $d=1$, $\mathrm{Def}_{D,\mathrm{Fil}^\bullet}$ is formally smooth. As in the characteristic 0 situation, the same is true for general d, so long as the parameter satisfies a certain regularity condition. Note that the regularity condition in here is slightly different than in characteristic 0; the additional characters avoided in the statement of [BC09, Proposition 2.3.10] do not make sense in characteristic p.

Proposition 2.2.3. *Suppose the parameter* $\underline{\delta}$ *of* $\mathrm{Fil}^\bullet D$ *satisfies the property that* $\delta_i\delta_j^{-1}\neq\chi_{\mathrm{cyc}}\circ\mathrm{Nm}_{K/\mathbf{Q}_p}$ *for any* $i<j$. *Then* $\mathrm{Def}_{D,\mathrm{Fil}^\bullet}$ *is formally smooth.*

Proof. The proof is essentially identical to that of [BC09, Proposition 2.3.10], but we sketch it here for the convenience of the reader. We proceed by induction on d; the case $d = 1$ follows from the proof of [Bel20, Proposition 5.1], so we assume the result for trianguline deformations of (φ, Γ)-modules of rank $d - 1$. Let $I \subset R'$ be a square-zero ideal. We need to prove that $\mathrm{Def}_{D,\mathrm{Fil}^\bullet}(R') \to \mathrm{Def}_{D,\mathrm{Fil}^\bullet}(R'/I)$ is surjective, so we may factor $R' \twoheadrightarrow R'/I$ into a series of small extensions and assume that I is principal and $I\mathfrak{m}_{R'} = 0$. By the inductive hypothesis, we may find a trianguline deformation D' of $\mathrm{Fil}^{d-1} D$ over $\Lambda_{R',\mathrm{rig},L}$. By twisting, we may assume that δ_d is trivial. Then we need to show that the natural map $H^1_{\varphi,\Gamma}(D') \to H^1_{\varphi,\Gamma}(\mathrm{Fil}^{d-1})$ is surjective. But the cokernel of this map is $H^2_{\varphi,\Gamma}(I \otimes_{R'/\mathfrak{m}_{R'}} \mathrm{Fil}^{d-1} D(\delta_d^{-1})) = I \otimes_{R'/\mathfrak{m}_{R'}} H^2_{\varphi,\Gamma}(\mathrm{Fil}^{d-1} D(\delta_d^{-1}))$, which is 0 by assumption and [Bel20, Corollary 4.11]. □

In order to build moduli spaces of trianguline (φ, Γ)-modules, we will use moduli spaces of characters, as in [Bel21, §2.3]. If G is a commutative p-adic Lie group and $G' \subset G$ is a compact subgroup such that G/G' is free and finitely generated, then we have $\widehat{G'} := \mathrm{Spa}\,\mathbf{Z}_p[\![G']\!]$ and the pseudorigid spaces $\widehat{G'}^{\mathrm{an}}$ and $\widehat{G}^{\mathrm{an}} := \mathrm{Spa}(\mathbf{Z}[G/G'], \mathbf{Z}) \times_{\mathbf{Z}} \widehat{G'}^{\mathrm{an}}$. If X is a pseudorigid space, we also have the pseudorigid space $\widehat{G}_X$, which represents the functor

$$Y \rightsquigarrow \mathrm{Hom}_{\mathrm{cts}}(G, \mathscr{O}(Y))$$

on the category of adic spaces over X.

In particular, if K is a finite extension of $\mathbf{Q}_p$, we will be interested in $\widehat{K^\times}^{\mathrm{an}}$ and $\widehat{(K^\times)^d}^{\mathrm{an}}$ for $d \geq 1$:

Definition 2.2.4. We let $\mathcal{T} := \widehat{K^\times}^{\mathrm{an}}$, and for any $d \geq 1$, we write $\mathcal{T}^d := \widehat{(K^\times)^d}^{\mathrm{an}}$.

We see that $\widehat{K^\times}^{\mathrm{an}} \cong \mathbf{G}_m^{\mathrm{ad}} \times_{\mathbf{Z}} \mathrm{Spa}\,\mathbf{Z}_p[\![\mathscr{O}_K^\times]\!]^{\mathrm{an}}$, and $\mathcal{T}^d \cong \mathbf{G}_m^{\mathrm{ad},d} \times_{\mathbf{Z}} \mathrm{Spa}\,\mathbf{Z}_p[\![(\mathscr{O}_K^\times)^d]\!]^{\mathrm{an}}$. Since $\mathscr{O}_K^\times$ is compact, $\mathrm{Spa}\,\mathbf{Z}_p[\![\mathscr{O}_K^\times]\!]^{\mathrm{an}}$ is a quasi-compact pseudorigid space; it has a finite cover $\{U_i := \mathrm{Spa}\,R_i\}$ by affinoid subspaces, and $\mathbf{G}_{m,U_i}$ is a rising union of relative annuli $C_{U_i,h} := \mathrm{Spa}\,R_i \left\langle u^h T, u^h T^{-1} \right\rangle$.

If $K = \mathbf{Q}_p$, then $\widehat{\mathbf{Q}_p^\times}^{\mathrm{an}}$ has connected components indexed by the elements of μ_{p-1}, each of which is isomorphic to $(\mathrm{Spa}\,\mathbf{Z}_p[\![\mathbf{Z}_p]\!])^{\mathrm{an}} \times \mathbf{G}_m^{\mathrm{ad}}$.

Remark 2.2.5. In the pseudorigid setting (unlike the classical rigid analytic setting), it is not true that $\widehat{G_1 \times G_2}^{\mathrm{an}} \cong \widehat{G_1}^{\mathrm{an}} \times \widehat{G_2}^{\mathrm{an}}$. Indeed, $\mathrm{Spa}\,\mathbf{Z}_p[\![T_1, T_2]\!]^{\mathrm{an}}$ consists of all valuations which do not vanish on all three of p, T_1, T_2. But

$$\mathrm{Spa}\,\mathbf{Z}_p[\![T_1]\!]^{\mathrm{an}} \times_{\mathbf{Z}_p} \mathrm{Spa}\,\mathbf{Z}_p[\![T_2]\!]^{\mathrm{an}}$$

also excludes valuations vanishing at both p and T_1 (or both p and T_2). In particular, $\mathcal{T}^d$ is *not* a product of copies of $\mathcal{T}$.

Definition 2.2.6. We say that a continuous character $\kappa : K^\times \to \mathscr{O}(X)^\times$ is *regular* if for all maximal points $x \in X$, the residual character $\kappa_x : K^\times \to k(x)^\times$ is not of the form

- $\alpha \mapsto \alpha^{-\mathbf{i}}$ or $\alpha \mapsto \alpha^{\mathbf{i+1}}|\alpha|$ for $\mathbf{i} \in \mathbf{Z}_{\geq 0}^{\mathrm{Hom}(K,k(x))}$ (if x is a characteristic 0 point), or
- trivial or $\chi_{\mathrm{cyc}} \circ \mathrm{Nm}_{K/\mathbf{Q}_p}$ (if x is a characteristic p point).

The space of *regular parameters* $\mathcal{T}^d_{\mathrm{reg}} \subset \mathcal{T}^d$ is the Zariski-open subspace whose X-points are given by parameters $\underline{\delta} : (K^\times)^d \to \mathscr{O}(X)^\times$ such that $\delta_i\delta_j^{-1} : K^\times \to \mathscr{O}(X)^\times$ is regular for all $j > i$.

Consider the functor $\mathcal{S}_d^\square$ on pseudorigid spaces defined via

$$X \rightsquigarrow \{(D, \mathrm{Fil}^\bullet D, \underline{\delta}, \underline{\nu})\}/\sim$$

where D is a trianguline (φ, Γ_K)-module with filtration $\mathrm{Fil}^\bullet D$ and regular parameter $\underline{\delta} \in \mathcal{T}^d_{\mathrm{reg}}$, and $\underline{\nu}$ is a sequence of trivializations $\nu_i : \mathrm{gr}^i D \xrightarrow{\sim} \Lambda_{X,\mathrm{rig},K}$. There is a natural transformation $\mathcal{S}_d^\square \to \mathcal{T}^d_{\mathrm{reg}}$ given on X-points by

$$(D, \mathrm{Fil}^\bullet D, \underline{\delta}, \underline{\nu}) \rightsquigarrow \underline{\delta}$$

Exactly as in [Che13, Théorème 3.3] and [HS16, Theorem 2.4], we have the following:

Proposition 2.2.7. *The functor $\mathcal{S}_d^\square$ is representable by a pseudorigid space, which we also denote $\mathcal{S}_d^\square$, and the morphism $\mathcal{S}_d^\square \to \mathcal{T}^d_{\mathrm{reg}}$ is smooth of relative dimension $\frac{d(d-1)}{2}[K : \mathbf{Q}_p]$.*

One proves by induction on d that if D is a trianguline (φ, Γ_K)-module over X with parameter $\underline{\delta} \in (\mathcal{T}_{\mathrm{reg}})^d$, then $H^1_{\varphi,\Gamma_K}(D)$ is a vector bundle over X of rank $d[K : \mathbf{Q}_p]$ (the regularity assumption ensures that $H^0_{\varphi,\Gamma_K}(D) = H^2_{\varphi,\Gamma_K}(D) = 0$). Now $\mathcal{S}_1^\square = \mathcal{T} = \mathcal{T}^1_{\mathrm{reg}}$, so $\mathcal{S}_1^\square$ is representable and is smooth of the correct dimension over $\mathcal{T}^1_{\mathrm{reg}}$. Then one may proceed by induction on d again, and construct $\mathcal{S}_d^\square$ as the moduli space of extensions of the universal (φ, Γ_K)-module of character type $\Lambda_{\mathcal{T},\mathrm{rig},K}(\delta_{\mathrm{univ}})$ by the universal object $D_{d-1,\mathrm{univ}}$ over $\mathcal{S}_{d-1}^\square$. For a specified regular parameter $\underline{\delta} = (\delta_1, \ldots, \delta_d) \in \mathcal{T}^d_{\mathrm{reg}}(X)$, the fiber $\mathcal{S}_d^\square|_{\underline{\delta}}$ is equal to $\mathrm{Ext}^1(\Lambda_{X,\mathrm{rig},K}(\delta_d), D_{d-1,\mathrm{univ}}|_{(\delta_1,\ldots,\delta_{d-1})}) = H^1_{\varphi,\Gamma_K}(D_{d-1,\mathrm{univ}}|_{(\delta_1,\ldots,\delta_{d-1})}(\delta_d^{-1}))$. This is a rank-$(d-1)$ vector bundle over X, and the claim follows.

We also introduce variants of $\mathcal{S}_d^\square$ with families of fixed determinant and weights. More precisely, suppose X is a pseudorigid space and we have a continuous character $\delta_{\mathrm{det}} : K^\times \to \mathscr{O}(X)^\times$ and a d-tuple of continuous characters $\underline{\kappa} := (\kappa_1, \ldots, \kappa_d) : \mathscr{O}_K^\times \to \mathscr{O}(X)^\times$. We say that δ_{det} and $\underline{\kappa}$ are *compatible* if $\delta_{\mathrm{det}}|_{\mathscr{O}_K^\times} = \kappa_1 \cdots \kappa_d$. If δ_{det} and $\underline{\kappa}$ are compatible, we consider the functors $\mathcal{S}_d^{\square,\delta_{\det}}$ and $\mathcal{S}_d^{\square,\delta_{\det},\underline{\kappa}}$ on pseudorigid spaces over X defined via

$$Y \rightsquigarrow \{(D, \operatorname{Fil}^\bullet D, \underline{\delta}) \in \mathcal{S}_d^{\square}(Y) \mid \delta_1 \cdots \delta_d = \delta_{\det}\}/\sim$$

and

$$Y \rightsquigarrow \{(D, \operatorname{Fil}^\bullet D, \underline{\delta}, \underline{\nu}) \in \mathcal{S}_d^{\square}(Y) \mid \delta_i|_{\mathscr{O}_K^\times} = \kappa_i \text{ for all } i, \delta_1 \cdots \delta_d = \delta_{\det}\}/\sim$$

Proposition 2.2.8. *The functor $\mathcal{S}_d^{\square,\delta_{\det},\underline{\kappa}}$ is representable by a pseudorigid space over X, which we also denote $\mathcal{S}_d^{\square,\delta_{\det},\underline{\kappa}}$, and the morphism $\mathcal{S}_d^{\square,\delta_{\det},\underline{\kappa}} \to X$ is smooth and surjective of relative dimension $\frac{d(d-1)}{2}[K:\mathbf{Q}_p] + d - 1$.*

Proof. Set $Y := \widehat{(\mathscr{O}_K^\times)^d}^{\mathrm{an}}$. Then there is a morphism $\mathcal{T}^d \to \mathbf{G}_{m,Y}$, given by $\underline{\delta} \mapsto \Big(\delta_1|_{\mathscr{O}_K^\times}, \ldots, \delta_d|_{\mathscr{O}_K^\times}, \delta_1(\varpi_K)\cdots\delta_d(\varpi_K)\Big)$, and it is smooth of relative dimension $d-1$. The choice of $\delta_{\det}$ and $\underline{\kappa}$ define a morphism $X \to \mathbf{G}_{m,Y}$, and we have a pullback square

$$\begin{array}{ccc} \mathcal{S}_d^{\square,\delta_{\det},\underline{\kappa}} & \longrightarrow & \mathcal{S}_d^{\square} \\ \big\downarrow & & \big\downarrow \\ X & \longrightarrow & \mathbf{G}_{m,Y} \end{array}$$

Then the result follows from Proposition 2.2.7. $\square$

Example 2.2.9. In the example of most interest to us, we will take $K = \mathbf{Q}_p$, $d = 2$, and $R = \mathbf{Z}_p[\![T_0]\!]$, where $T_0 := \mathrm{T}(\mathbf{Z}_p)$ for a split maximal torus $\mathrm{T} \subset \mathrm{GL}_2/\mathbf{Z}_p$. Fix an unramified character $\psi_0 : \mathrm{Gal}_{\mathbf{Q}_p} \to R^\times$. There is a universal pair of characters $\lambda_1, \lambda_2 : \mathbf{Z}_p^\times \rightrightarrows R^\times$, and we set $\psi := (\lambda_1\lambda_2\chi_{\mathrm{cyc}})^{-1}\psi_0$ and $\underline{\kappa} : (\lambda_2^{-1}, (\lambda_1\chi_{\mathrm{cyc}})^{-1})$. Then the morphism $\mathcal{S}_2^{\square} \to \operatorname{Spa} R^{\mathrm{an}}$ is the natural projection $\mathcal{S}_d^{\square} \to \widehat{(\mathbf{Z}_p^\times)^2}$, composed with taking inverses and swapping factors. Furthermore, $\mathcal{T}$ is 2-dimensional and irreducible (corresponding to a choice of δ_1); fixing the determinant means the remaining degrees of freedom are the 1-dimensional irreducible space $\widehat{\mathbf{Z}_p^\times}$ (corresponding to the choice of $\delta_2|_{\mathbf{Z}_p^\times}$), and the generically 1-dimensional space of extensions between them. We see that in this case, $\mathcal{S}_2^{\square,\delta_\psi,\underline{\kappa}}$ is 4-dimensional, and an $\mathbf{A}^1$-torsor over a dense open subspace of $\mathbf{G}_m^{\mathrm{ad}} \times \widehat{(\mathbf{Z}_p^\times)^2}$. Hence it is irreducible.

2.3. **Structure of trianguline varieties.** Let $K/\mathbf{Q}_p$ be a finite extension, and let $\overline{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_d(k)$ be a continuous representation, where k is a finite field containing the residue field of K. The fiber product $(\operatorname{Spa} R_{\overline{\rho}}^{\square})^{\mathrm{an}} \times_{\operatorname{Spa}\mathbf{Z}_p} \mathcal{T}^d$ exists as a pseudorigid space, and it is contained in the fiber product

$$\mathbf{G}_m^{\mathrm{ad},d} \times_{\mathbf{Z}} \widehat{(\mathscr{O}_K^\times)^d} \times \operatorname{Spa}(R_{\overline{\rho}}^{\square})^{\mathrm{an}} \subset \mathbf{G}_m^{\mathrm{ad},d} \times_{\operatorname{Spa}\mathbf{Z}} \operatorname{Spa}(R_{\overline{\rho}}^{\square} \widehat{\otimes} \mathbf{Z}_p[\![(\mathscr{O}_K^\times)^d]\!])^{\mathrm{an}}$$

(with complement of codimension ≥ 2 if $d \geq 2$). Let $X^{\square}_{\mathrm{tri},\overline{\rho}}$ be the Zariski closure in the latter of the set of maximal points $x = \{(\rho_x, \underline{\delta}_x)\}$, where ρ_x is a (framed) lift of $\overline{\rho}$ and $\underline{\delta}_x \in \mathcal{T}^d_{\mathrm{reg}}(L)$ is a regular parameter of $D_{\mathrm{rig}}(\rho_x)$.

Let R be a complete local noetherian $\mathbf{Z}_p$-algebra with finite residue field. Fix an d-tuple of characters $\underline{\kappa} := (\kappa_1, \ldots, \kappa_d)$, where $\kappa_i : \mathscr{O}_K^\times \to \mathscr{O}(X)^\times$ and $X := (\operatorname{Spa} R)^{\mathrm{an}}$, and fix a character $\psi : \mathrm{Gal}_K \to R^\times$. Over the pseudorigid space X, a character $\psi : \mathrm{Gal}_K \to \mathscr{O}(X)^\times$ corresponds to a rank-1 (φ, Γ)-module of the form $D_{\mathrm{rig}}(\delta_\psi)$, for some character $\delta_\psi : K^\times \to \mathscr{O}(X)^\times$. If δ_ψ and $\underline{\kappa}$ are compatible, we may define $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho}} \subset (\operatorname{Spa} R^{\square,\psi}_{\overline{\rho}})^{\mathrm{an}} \times \mathcal{T}^d$ to be the Zariski closure of the set of maximal points $x = \{(\rho_x, \underline{\delta}_x)\}$, where ρ_x is a framed lift of $\overline{\rho}$ with determinant ψ and $\underline{\delta}_x \in \mathcal{T}^d_{\mathrm{reg}}(L)$ is a regular parameter of $D_{\mathrm{rig}}(\rho_x)$ such that $\delta_i|_{\mathscr{O}_K^\times} = \kappa_i$.

In order to study the structure of $X^{\square}_{\mathrm{tri},\overline{\rho}}$ and $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho}}$, we will need to know something about the essential image of the functor from Galois representations to (φ, Γ)-modules. We refer the reader to [Bel20] for details on definitions of pseudorigid overconvergent period rings and the construction of (φ, Γ)-modules in the pseudorigid setting. However, we note here that $\Lambda_{R,[0,b],K}$ is the coordinate ring of a closed annulus over $\operatorname{Spa} R$, $\Lambda_{R,(0,b],K}$ is the ring of global functions on a half-open annulus over $\operatorname{Spa} R$, and $\Lambda_{R,\mathrm{rig},K} := \varinjlim_{b\to 0} \Lambda_{R,(0,b],K}$. As in the work of [CC98] and [BC08], (φ, Γ)-modules attached to Galois representations are constructed over $\Lambda_{R,[0,b],K}$ for some $b > 0$ (which depends in subtle ways on the representation).

Lemma 2.3.1. *The functor $M \rightsquigarrow D_{\mathrm{rig},K}(M)$ from Gal_K-representations to their associated (φ, Γ)-modules is formally smooth.*

Proof. We need to show that if D is a projective (φ, Γ_K)-module over a pseudoaffinoid algebra R', and $I \subset R'$ is a square-zero ideal such that $(R'/I) \otimes_{R'} D$ arises from a family of Galois representations, then D also arises from a family of Galois representations. Indeed, we have a short exact sequence

$$0 \to ID \to D \to (R'/I) \otimes_{R'} D \to 0$$

By assumption, $D' := (R'/I) \otimes_{R'} D$ arises from a family of Gal_K representations M' over R'/I, and since

$$D'' := ID \cong I \otimes_{R'} D \cong (R'/\operatorname{ann}_{R'} I) \otimes_{R'/I} D$$

it arises from a family of Gal_K representations M'' over $R'/\operatorname{ann}_{R'} I$. Since D has a model D_b over $\Lambda_{R',(0,b],K}$, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\Lambda}_{R',(0,b/p]} \otimes_{R'} D'' \arrow[r] & \widetilde{\Lambda}_{R',(0,b/p]} \otimes_{R'} D \arrow[r] & \widetilde{\Lambda}_{R',(0,b/p]} \otimes_{R'} D' \arrow[r] & 0 \\ 0 \arrow[r] & \widetilde{\Lambda}_{R',(0,b]} \otimes_{R'} D'' \arrow[r] \arrow[u, "\varphi-1"] & \widetilde{\Lambda}_{R',(0,b]} \otimes_{R'} D \arrow[r] \arrow[u, "\varphi-1"] & \widetilde{\Lambda}_{R',(0,b]} \otimes_{R'} D' \arrow[r] \arrow[u, "\varphi-1"] & 0 \end{tikzcd}$$

By construction, $\widetilde{\Lambda}_{R',(0,b]}\otimes_{R'}D'' \cong \widetilde{\Lambda}_{R',(0,b]}\otimes\left(\widetilde{\Lambda}_{R'_0,[0,b]}\otimes_{R'_0} M''_0\right)$ and $\widetilde{\Lambda}_{R',(0,b]}\otimes_{R'} D' \cong \widetilde{\Lambda}_{R',(0,b]}\otimes\left(\widetilde{\Lambda}_{R'_0,[0,b]}\otimes_{R'_0} M'_0\right)$, for some integral models M''_0 and M'_0 (perhaps after localizing on $\operatorname{Spa} R'$ and shrinking b). Therefore, we have quasi-isomorphisms

$$\begin{aligned}[M''] &\xrightarrow{\sim} [\widetilde{\Lambda}_{R',[0,b]}\otimes_{R'_0} M''_0 \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R',[0,b/p]}\otimes_{R'_0} M''_0] \\ &\xrightarrow{\sim} [\widetilde{\Lambda}_{R',(0,b]}\otimes_{R'} D'' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R',(0,b/p]}\otimes_{R'} D'']\end{aligned}$$

and

$$\begin{aligned}[M'] &\xrightarrow{\sim} [\widetilde{\Lambda}_{R',[0,b]}\otimes_{R'_0} M'_0 \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R',[0,b/p]}\otimes_{R'_0} M'_0] \\ &\xrightarrow{\sim} [\widetilde{\Lambda}_{R',(0,b]}\otimes_{R'} D' \xrightarrow{\varphi-1} \widetilde{\Lambda}_{R',(0,b/p]}\otimes_{R'} D']\end{aligned}$$

Then the snake lemma implies that we have an exact sequence

$$0\to M''\to \left(\widetilde{\Lambda}_{R',\mathrm{rig},K}\otimes D\right)^{\varphi=1}\to M'\to 0$$

of R'-modules equipped with continuous R'-linear actions of Gal_K, with M' finite projective over R'/I and $M''\cong (R'/\operatorname{ann}_{R'} I)\otimes_{R'/I} M'$. It follows that $M := \left(\widetilde{\Lambda}_{R',\mathrm{rig},K}\otimes D\right)^{\varphi=1}$ is a projective R'-module of the same rank and $D_{\mathrm{rig},K}(M) = D$. $\square$

In [BHS17, §2.2], the authors show that the characteristic 0 locus $X_{\mathrm{tri},\overline{\rho}}^{\square,\mathrm{rig}}$ of the trianguline variety is equidimensional of dimension $d^2+[K:\mathbf{Q}_p]\frac{d(d+1)}{2}$, and generically smooth. We note that if $\psi:\mathrm{Gal}_K\to\mathscr{O}_E^\times$ is a crystalline character, where $E/\mathbf{Q}_p$ is a finite extension and $\mathscr{O}_E$ is its ring of integers, then an identical argument shows that the rigid analytic locus $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\mathrm{rig}}\subset X_{\mathrm{tri},\overline{\rho}}^{\square,\psi}$ is equidimensional of dimension $d^2-1+[K:\mathbf{Q}_p]\frac{(d+2)(d-1)}{2}$ (indeed, [BIP22, Theorem 1.2] provides the necessary crystalline lifts with fixed determinant).

Unfortunately, we cannot rule out components of $X_{\mathrm{tri},\overline{\rho}}^{\square}$ or $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi}$ supported entirely in characteristic p, and so to deduce the same result in the pseudo-rigid setting, we need to repeat a large part of the argument in a neighborhood of the characteristic p fiber.

Proposition 2.3.2. (1) *The space $X_{\mathrm{tri},\overline{\rho}}^{\square}$ (equipped with its underlying reduced structure) is equidimensional of dimension $d^2+[K:\mathbf{Q}_p]\frac{d(d+1)}{2}$.*

(2) *If $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}}$ is non-empty, it is equidimensional of dimension $d^2-1+[K:\mathbf{Q}_p]\frac{d(d-1)}{2}+\dim\operatorname{Spa}R^{\mathrm{an}}$.*

(3) *There is an open subspace $Z\subset\operatorname{Spa}R^{\mathrm{an}}$ such that morphism $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}}|_Z\to Z$ is equidimensional of dimension $d^2-1+[K:\mathbf{Q}_p]\frac{d(d-1)}{2}$.*

Proof. The proofs of the first two parts are very similar to that of [BHS17, Théorème 2.6], and we will prove them simultaneously. By construction, there is a universal framed deformation $\rho_{\mathrm{univ}} : \mathrm{Gal}_K \to \mathrm{GL}_d(R_{\overline{\rho}}^{\square})$ of $\overline{\rho}$, and we may pull it back to $X_{\mathrm{tri},\overline{\rho}}^{\square}$ (resp. $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}}$). Then for any irreducible open affinoid $X \subset X \to X_{\mathrm{tri},\overline{\rho}}^{\square}$ (resp. $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}}$), by [Bel21, Corollary 5.10] there is a sequence of blow-ups and normalizations $f : \widetilde{X} \to X$ and an open subspace $U \subset \widetilde{X}$ containing the characteristic p locus such that $f^*\rho_{\mathrm{univ}}|_U$ is trianguline with parameters $f^*\underline{\delta}$. Shrinking U if necessary, we may assume that $f^*\underline{\delta}$ is regular (indeed, the pre-image of $\mathcal{T}_{\mathrm{reg}}^d$ in U is open, and by construction U contains a Zariski dense set of points corresponding to trianguline representations with regular parameters). Furthermore, there is a Zariski-dense and open subspace $V \subset X_{\mathrm{tri},\overline{\rho}}^{\square}$ (resp. $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}}$) such that $f^{-1}(V) \subset U$ and f defines an isomorphism $f^{-1}(V) \xrightarrow{\sim} V$.

Over U, the (φ, Γ_K)-module $D := D_{\mathrm{rig},K}(f^*\rho_{\mathrm{univ}})$ is equipped with an increasing filtration $\mathrm{Fil}^\bullet D$ such that $\mathrm{gr}^i D \cong \Lambda_{U,\mathrm{rig},K}(f^*\delta_i) \otimes \mathscr{L}_i$ for some line bundle $\mathscr{L}_i$ on U. We may therefore construct a $\mathbf{G}_{m,U}^d$-torsor $U^\square \to U$ trivializing each of the $\mathscr{L}_i$; since $U^\square$ carries the data $(D, \mathrm{Fil}^\bullet D, f^*\underline{\delta}, \underline{\nu})$, where $\underline{\nu}$ is the set of trivializations $\nu_i : \mathrm{gr}^i D \xrightarrow{\sim} \Lambda_{U,\mathrm{rig},K}(f^*\delta_i)$, there is a morphism $U^\square \to \mathcal{S}_d^\square$.

Let $V^\square \subset U^\square$ denote the pullback of $U^\square \to U$ to V. We claim that $V^\square \to \mathcal{S}_d^\square$ is smooth of relative dimension d^2. To see this, suppose we have a pseudoaffinoid algebra R', a morphism $\mathrm{Spa}\, R' \to \mathcal{S}_d^\square$, and a square-zero ideal $I \subset R'$ such that the composition $\mathrm{Spa}\, R'/I \hookrightarrow \mathrm{Spa}\, R' \to \mathcal{S}_d^\square$ is in the image of $V^\square$. Then there is a ring of definition $R_0' \subset R'/I$ such that the homomorphism $R_{\overline{\rho}}^\square \to R'/I$ factors through R_0'; we let $M_0' \cong R_0'^{\oplus d}$ be the pullback of the universal framed deformation to R_0' and we let $M' := R'/I \otimes_{R_0'} M_0'$.

By Lemma 2.3.1, there is a Gal_K-representation M over R' such that $(R'/I) \otimes_{R'} M \xrightarrow{\sim} M'$. It follows that M_0' and its basis lift to a free module M_0 over some ring of definition $R_0' \subset R'$, such that $R' \otimes_{R_0'} M_0 = M$. Moreover, M' is residually a lift of $\overline{\rho}$ at every maximal point of $\mathrm{Spa}\, R'$, so M is as well. By [WE18, Theorem 3.8], M_0 corresponds to a $\mathrm{Spa}\, R_0'$-point of $\mathrm{Spf}\, R_{\overline{\rho}}^\square$, and by construction M corresponds to a $\mathrm{Spa}\, R'$-point of $X_{\mathrm{tri},\overline{\rho}}^\square$ deforming M'. Since M' corresponds to a $\mathrm{Spa}(R'/I)$-point of the Zariski-open subspace $V \subset X_{\mathrm{tri},\overline{\rho}}^\square$, the image of the morphism $\mathrm{Spa}\, R' \to X_{\mathrm{tri},\overline{\rho}}^\square$ also lands in V. Since D is trianguline with regular parameters and trivialized quotients, the morphism $\mathrm{Spa}\, R' \to V$ lifts to a morphism $\mathrm{Spa}\, R' \to V^\square$.

The claim that $V^\square \to \mathcal{S}_d^\square$ has relative dimension d^2 follows because "changing the framing" makes $V^\square$ a $(\mathrm{GL}_d)^{\mathrm{an}}$-torsor over its image in $\mathcal{S}_d^\square$.

Now we can compute the dimension. By Proposition 2.2.7, we see that $V^\square$ is equidimensional of dimension $d^2 + \frac{d(d-1)}{2}[K:\mathbf{Q}_p] + d[K:\mathbf{Q}_p] + d$ (resp. $d^2+\frac{d(d-1)}{2}[K:\mathbf{Q}_p]+d[K:\mathbf{Q}_p]+d+\dim \operatorname{Spa} R^{\mathrm{an}}$). Since $V^\square \to V$ is a $\mathbf{G}_{m,V}^d$-torsor, it follows that V is equidimensional of dimension $d^2 + \frac{d(d+1)}{2}[K:\mathbf{Q}_p]$ (resp. $d^2 + \frac{d(d-1)}{2}[K:\mathbf{Q}_p] + d[K:\mathbf{Q}_p] + \dim \operatorname{Spa} R^{\mathrm{an}}$. Finally, $V \subset X$ is Zariski-dense, so we are done.

For the last part, we define $V^{\square,\psi,\underline{\kappa}}$ via the pullback

$$\begin{tikzcd}
V^{\square,\psi,\underline{\kappa}} \arrow[r] \arrow[d] & V^\square \arrow[d] \\
\mathcal{S}_d^{\square,\delta_\psi,\underline{\kappa}} \arrow[r] \arrow[d] & \mathcal{S}_d^\square \arrow[d] \\
\operatorname{Spa} R^{\mathrm{an}} \arrow[r] & \mathbf{G}_{m,Y}
\end{tikzcd}$$

where $Y := \widehat{(\mathscr{O}_K^\times)^d}$ and the morphism $\operatorname{Spa} R^{\mathrm{an}} \to \mathbf{G}_{m,Y}$ is given by $\underline{\kappa}$ and δ_ψ. Since $V^\square \to \mathbf{G}_{m,Y}$ is smooth, its image is open, and the pre-image in $\operatorname{Spa} R^{\mathrm{an}}$ is open, as well. $\square$

Remark 2.3.3. Suppose that $x \in \operatorname{Spa} R^{\mathrm{an}}$ is a maximal point such that the fiber of $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}}$ contains a point $(\rho_x, \underline{\delta_x})$ such that $\underline{\delta_x}$ is a regular parameter for $D_{\mathrm{rig}}(\rho_x)$. Then if we apply Proposition 2.3.2 with $R = k(x)^+$, we see that every irreducible component of the fiber containing $(\rho_x, \underline{\delta_x})$ has dimension $d^2 + \frac{d(d-1)}{2}[K:\mathbf{Q}_p] + d[K:\mathbf{Q}_p]$.

Example 2.3.4. We return to the setting of Example 2.2.9, where $K = \mathbf{Q}_p$, $d = 2$, $R = \mathbf{Z}_p[\![T_0]\!]$ corresponds to integral weight space for a split maximal torus of $\mathrm{GL}_2/\mathbf{Z}_p$, $\psi_0 : \mathrm{Gal}_{\mathbf{Q}_p} \to R^\times$ is an unramified character, and there is a universal pair of characters $\lambda_1, \lambda_2 : \mathbf{Z}_p^\times \rightrightarrows R^\times$. We again set $\psi := \psi_0\,(\lambda_1\lambda_2\chi_{\mathrm{cyc}})^{-1}$ and $\underline{\kappa} : (\lambda_2^{-1}, (\lambda_1\chi_{\mathrm{cyc}})^{-1})$. Then if $X_{\mathrm{tri},\overline{\rho}}^{\square,\psi,\underline{\kappa}}$ is non-empty, each irreducible component is 6-dimensional.

Moreover, suppose there is a characteristic-p point $(\rho_x, \underline{\delta_x})$ with specified weight and determinant, such that ρ_x is trianguline with regular parameter δ_x. Then the fiber over $\underline{\delta_x}|_{(\mathbf{Z}_p^\times)^2}$ is 4-dimensional; since this is one of $p-1$ disjoint characteristic-p fibers, we see that the irreducible component containing $(\rho_x, \underline{\delta_x})$ contains a dense open characteristic-0 subspace, consisting of points in $U_{\mathrm{tri}}^\square(\overline{\rho})^{\mathrm{reg}}$ (in the notation of [BHS17, Définition 2.4]).

Now we consider a global setup. Let F be a number field, and suppose that $\overline{\rho} : \mathrm{Gal}_F \to \mathrm{GL}_d(\mathbf{F})$ is an absolutely irreducible representation, unramified outside a finite set of primes S.

Then the homomorphisms
$$R_{\overline{\rho}_v}^\square \to R_{\overline{\rho},S}^\square$$

for each $v \mid p$ induce a morphism

$$\left(\operatorname{Spa} R^{\square}_{\overline{\rho},S}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d \to \prod_{v|p} \left(\left(\operatorname{Spa} R^{\square}_{\overline{\rho}_v}\right)^{\mathrm{an}} \times \mathcal{T}^d\right)$$

and we define $X^{\square}_{\mathrm{tri},\overline{\rho},S}$ to be the pre-image of $\prod_{v|p} X^{\square}_{\mathrm{tri},\overline{\rho}_v}$.

If R is a complete local noetherian $\mathbf{Z}_p$-algebra with maximal ideal $\mathfrak{m}_R$ and finite residue field, and $\psi : \mathrm{Gal}_F \to R^\times$ is a continuous character such that $\det \overline{\rho} = \psi \mod \mathfrak{m}_R$, the homomorphisms

$$R^{\square,\psi_v}_{\overline{\rho}_v} \to R^{\square,\psi}_{\overline{\rho},S}$$

and

$$R^{\square,\psi}_{\overline{\rho},\mathrm{loc}} \to R^{\square,\psi}_{\overline{\rho},S}$$

induce a sequence of morphisms

$$\left(\operatorname{Spa} R^{\square,\psi}_{\overline{\rho},S}\right)^{\mathrm{an}} \times \textstyle\prod_{v|p} \mathcal{T}^d \longrightarrow \left(\operatorname{Spa} R^{\square,\psi}_{\overline{\rho},\mathrm{loc}}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d \hookrightarrow \prod_{v|p} \left(\left(\operatorname{Spa} R^{\square,\psi_v}_{\overline{\rho}_v}\right)^{\mathrm{an}} \times \mathcal{T}^d\right)$$

where $\psi_v := \psi|_{\mathrm{Gal}_{F_v}}$. We define $X^{\square,\psi}_{\mathrm{tri},\overline{\rho},S}$ and $X^{\square,\psi}_{\mathrm{tri},\overline{\rho},\mathrm{loc}}$ to be the pre-images of $\prod_{v|p} X^{\square,\psi_v}_{\mathrm{tri},\overline{\rho}_v}$ in $\left(\operatorname{Spa} R^{\square,\psi}_{\overline{\rho},S}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d$ and $\left(\operatorname{Spa} R^{\square,\psi}_{\overline{\rho},\mathrm{loc}}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d$, respectively.

If we additionally have d-tuples of characters $\underline{\kappa}_v := (\kappa_{v,1}, \ldots, \kappa_{v,d})$, where $\kappa_{v,i} : \mathscr{O}_{F_v}^\times \to \mathscr{O}(X)^\times$ is a continuous character, and we set $X := (\operatorname{Spa} R)^{\mathrm{an}}$, we may form the spaces

$$\begin{array}{ccccc}
X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},S} & \longrightarrow & X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}} & \hookrightarrow & \prod_{v|p} X^{\square,\psi_v,\underline{\kappa}_v}_{\mathrm{tri},\overline{\rho}_v} \\
\cap & & \cap & & \cap \\
\left(\operatorname{Spa} R^{\square,\psi}_{\overline{\rho},S}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d & \longrightarrow & \left(\operatorname{Spa} R^{\square,\psi}_{\overline{\rho},\mathrm{loc}}\right)^{\mathrm{an}} \times \prod_{v|p} \mathcal{T}^d & \hookrightarrow & \prod_{v|p} \left(\left(\operatorname{Spa} R^{\square,\psi_v}_{\overline{\rho}_v}\right)^{\mathrm{an}} \times \mathcal{T}^d\right)
\end{array}$$

In particular, suppose we have fixed a neat level $K = K^p I$, as in sections 3 and 4, and consider the ring $R = \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]$ corresponding to integral weight space. Since $T_0 = \prod_{v|p} (\operatorname{Res}_{\mathscr{O}_{F_v}/\mathbf{Z}_p} T_v)(\mathbf{Z}_p)$, we have homomorphisms $\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!] \to R$, and hence morphisms $\operatorname{Spa} R \to \operatorname{Spa} \mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]$. Suppose we have a determinant character $\psi : \mathrm{Gal}_F \to R^\times$ and a set of weights $\underline{\kappa}_v := (\kappa_{v,1}, \ldots, \kappa_{v,d}) : \mathscr{O}_{F_v}^\times \to \mathscr{O}(\mathscr{W}_F)^\times$ for each $v \mid p$, such that $\psi|_{\mathrm{Gal}_{F_v}}$ and $\underline{\kappa}_v$ are compatible for all v, and such that ψ_v and $\underline{\kappa}_v$ factor through $\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!] \to R$ for all v, i.e., they depend only on the projection to $\operatorname{Spa} \mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]$.

Proposition 2.3.5. *Under the above assumptions, there is an open subspace $Z \subset \mathscr{W}_F$ such that $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}}|_Z \to Z$ is equidimensional of dimension $|\Sigma_p|(d^2 - 1) + [F : \mathbf{Q}]\frac{d(d-1)}{2}$.*

Proof. Viewing ψ_v as a character $\mathrm{Gal}_{F_v} \to \mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]^\times$ and viewing $\underline{\kappa}_v = (\kappa_{v,1}, \ldots, \kappa_{v,d})$ as a d-tuple of characters $\mathscr{O}_{F_v}^\times \to \mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]^\times$, we have a pullback diagram

$$\begin{array}{ccc} X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}} & \longrightarrow & \prod_{v|p} X^{\square,\psi_v,\underline{\kappa}_v}_{\mathrm{tri},\overline{\rho}_v} \\ \downarrow & & \downarrow \\ \mathscr{W}_F & \longrightarrow & \prod_{v|p} (\mathrm{Spa}\,\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!])^{\mathrm{an}} \end{array}$$

The right vertical morphism has relative dimension

$$\sum_{v|p}\left(d^2 - 1 + [F_v : \mathbf{Q}_p]\frac{d(d-1)}{2}\right) = |\Sigma_p|(d^2-1) + [F : \mathbf{Q}]\frac{d(d-1)}{2}$$

over an open subspace of $\prod_{v|p} (\mathrm{Spa}\,\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!])^{\mathrm{an}}$, so the morphism $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}} \to \mathscr{W}_F$ does, as well. $\square$

The case we will be most interested in is the case where $F/\mathbf{Q}$ is cyclic and totally split at p, and $d = 2$. In that case, $X^{\square,\psi_v,\underline{\kappa}_v}_{\mathrm{tri},\overline{\rho}_v} \to \mathrm{Spa}\,\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]^{\mathrm{an}}$ has relative dimension 4 over an open subspace of $\mathrm{Spa}\,\mathbf{Z}_p[\![T_v(\mathscr{O}_{F_v})]\!]^{\mathrm{an}}$ for each $v \mid p$, and hence $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}} \to \mathscr{W}_F$ has relative dimension $4[F : \mathbf{Q}]$ over an open subspace of $\mathscr{W}_F$.

2.4. **Trianguline deformation rings.** We have constructed the trianguline varieties $X^{\square}_{\mathrm{tri},\overline{\rho}}$ and $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho}}$ as subspaces of the (non-quasicompact) pseudorigid space $\left(\mathrm{Spa}\,R^{\square}_{\overline{\rho}}\right)^{\mathrm{an}} \times \mathscr{T}^d$. However, the advantage of working with general pseudorigid spaces is that we can construct integral models, so long as we bound the slope.

We will apply this to find formal models for pieces of our trianguline varieties. Recall that when K is a finite extension of $\mathbf{Q}_p$ and $\overline{\rho}$ is a representation of Gal_K, we defined $X^{\square}_{\mathrm{tri},\overline{\rho}}$ and $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho}}$ as analytic subspaces of $\mathbf{G}_m^{\mathrm{ad}} \times \left(\mathrm{Spa}\,R^{\square}_{\overline{\rho}} \widehat{\otimes} \mathbf{Z}_p[\![(\mathscr{O}_K^\times)^d]\!]\right)^{\mathrm{an}}$ and $\mathbf{G}_m^{\mathrm{ad}} \times \left(\mathrm{Spa}\,R \widehat{\otimes} R^{\square}_{\overline{\rho}} \widehat{\otimes} \mathbf{Z}_p[\![(\mathscr{O}_K^\times)^d]\!]\right)^{\mathrm{an}}$, respectively. By construction, $\left(\mathrm{Spa}\,R^{\square}_{\overline{\rho}} \widehat{\otimes} \mathbf{Z}_p[\![(\mathscr{O}_K^\times)^d]\!]\right)^{\mathrm{an}}$ has an integral model, but $\mathbf{G}_m^{\mathrm{ad}} \times \left(\mathrm{Spa}\,R^{\square}_{\overline{\rho}} \widehat{\otimes} \mathbf{Z}_p[\![(\mathscr{O}_K^\times)^d]\!]\right)^{\mathrm{an}}$ does not; in particular, it is not equal to the analytic locus of $\mathrm{Spa}\,R^{\square}_{\overline{\rho}} \widehat{\otimes} \mathbf{Z}_p[\![(\mathscr{O}_K^\times)^d]\!]\left\langle T, T^{-1}\right\rangle$ (and similarly for $\mathbf{G}_m^{\mathrm{ad}} \times \left(\mathrm{Spa}\,R \widehat{\otimes} R^{\square}_{\overline{\rho}} \widehat{\otimes} \mathbf{Z}_p[\![(\mathscr{O}_K^\times)^d]\!]\right)^{\mathrm{an}}$).

However, let $Y := \widehat{\mathscr{O}_K^\times}$; after choosing a basis for $\mathscr{O}_K$ over $\mathbf{Z}_p$ and corresponding coordinates on Y, we may consider relative annuli $C_{Y,h}$, so that $\mathbf{G}^{\mathrm{ad}}_{m,Y} \cong \cup_h C_{Y,h}$.

In the particular case $K = \mathbf{Q}_p$, we may write $Y = \coprod_{a\in\mu_{p-1}} \mathrm{Spa}(\mathbf{Z}_p[\![u]\!])^{\mathrm{an}}$, where $u = [\gamma] - 1$ for some topological generator of $1 + p\mathbf{Z}_p$. We may cover $\mathrm{Spa}(\mathbf{Z}_p[\![u]\!]^{\mathrm{an}}$ with the open affinoid subspaces $U_1 := \mathrm{Spa}\left(\mathbf{Q}_p\left\langle\frac{u}{p}\right\rangle\right)$ and $U_2 := \mathrm{Spa}\left(\mathbf{Z}_p[\![u]\!]\left\langle\frac{p}{u}\right\rangle\left[\frac{1}{u}\right]\right)$; their intersection is the circle $U_1 \cap U_2 = \mathrm{Spa}\left(\mathbf{Q}_p\left\langle\frac{u}{p},\frac{p}{u}\right\rangle\right)$.

The annulus $C_{U_1,h}$ is affinoid, with coordinate ring

$$\mathbf{Q}_p\left\langle\frac{u}{p}, p^hT, T_2\right\rangle/(TT_2 - p^h) = \mathbf{Q}_p\left\langle\frac{u}{p}, T, T_1, T_2\right\rangle/(T_1 - p^hT, TT_2 - p^h)$$

Restricting to $U_1 \cap U_2$, we obtain an affinoid with coordinate ring

$$\mathbf{Z}_p[\![u]\!]\left\langle\frac{u}{p},\frac{p}{u}, T, T_1, T_2\right\rangle\left[\frac{1}{u}\right]/(T_1 - u^h\left(\frac{p}{u}\right)^h T, TT_2 - u^h\left(\frac{p}{u}\right)^h)$$

Writing $T_1' := \left(\frac{u}{p}\right)^h T_1$ and $T_2' := \left(\frac{u}{p}\right)^h T_2$, we get

$$\mathbf{Z}_p[\![u]\!]\left\langle\frac{u}{p},\frac{p}{u}, T, T_1', T_2'\right\rangle\left[\frac{1}{u}\right]/(T_1' - u^hT, TT_2' - u^h)$$

which is also the restriction of $C_{U_2,h}$ to $U_1 \cap U_2$.

Thus, we see that $C_{Y,h}$ in this case is the disjoint union of copies of

$$\mathrm{Spa}\left(\mathbf{Z}_p[\![u]\!]\langle T, T_1, T_2, T_1', T_2'\rangle/\left(p^hT_1' - u^hT_1, p^hT_2' - u^hT_2, T_1 - p^hT, T_1T_2 - p^h, T_1' - u^hT, TT_2' - u^h\right)\right)^{\mathrm{a}}$$

Returning to the general case, we may let $\mathfrak{T}_h$ be an integral model of $C_{Y,h}$ as above; if $\mathscr{O}_K^\times \cong G \times \mathbf{Z}_p^{\oplus r}$ as a topological group (with G a finite discrete group), then we see that $\mathfrak{T}_h$ can be cut out of a disjoint union of copies of $\mathrm{Spa}\,\mathbf{Z}_p[\![u_1,\ldots,u_r]\!]\langle T, \{T_{1,i}, T_{2,i}\}_{i=0}^r\rangle$. We let $\mathfrak{T}_h^\circ \subset \mathfrak{T}_h$ be an integral model of a relative open annulus, i.e., cut out of copies of $\mathrm{Spa}\,\mathbf{Z}_p[\![T, \{u_i, T_{1,i}, T_{2,i}\}_{i=0}^r]\!]$ instead.

Now we set

$$X^{\square}_{\mathrm{tri},\overline{\rho},h} := X^{\square}_{\mathrm{tri},\overline{\rho}} \cap \left((\mathfrak{T}_h^\circ)^d \times \mathrm{Spa}\, R^{\square}_{\overline{\rho}}\right)^{\mathrm{an}}$$

and

$$X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},h} := X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho}} \cap \left((\mathfrak{T}_d^\circ)^d \times \mathrm{Spa}\, R\,\widehat{\otimes}\, R^{\square}_{\overline{\rho}}\right)^{\mathrm{an}}$$

When F is a totally real field and $\overline{\rho}$ is a representation of Gal_F unramified outside a finite set of places S, we may similarly define bounded global trianguline varieties $X^{\square}_{\mathrm{tri},\overline{\rho},S,h}$ and $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},S,h}$ as subspaces of $\left(\prod_{v|p}(\mathfrak{T}_h^\circ)^d \times \mathrm{Spa}\, R^{\square}_{\overline{\rho},S}\right)^{\mathrm{an}}$ and $\left(\prod_{v|p}(\mathfrak{T}_h^\circ)^d \times \mathrm{Spa}\, R\,\widehat{\otimes}\, R^{\square}_{\overline{\rho},S}\right)^{\mathrm{an}}$, respectively.

Thus, we may apply Proposition A.0.2 to construct integral models of pieces of trianguline varieties:

Corollary 2.4.1. *Suppose that $\overline{\rho}$ is a representation of Gal_K, where K is a finite extension of $\mathbf{Q}_p$, or of Gal_F, where F is a totally real number field (in which case we assume ρ is unramified outside a finite set of places S). Then there are affine formal schemes $\mathfrak{X}^{\square}_{\mathrm{tri},\overline{\rho},h} = \mathrm{Spf}\, R^{\square}_{\mathrm{tri},\overline{\rho},h}$ (resp. $\mathfrak{X}^{\square}_{\mathrm{tri},\overline{\rho},S,h} = \mathrm{Spf}\, R^{\square}_{\mathrm{tri},\overline{\rho},S,h}$) and $\mathfrak{X}^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},h}$ (resp. $\mathfrak{X}^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},S,h}$) such that $\left(\mathfrak{X}^{\square}_{\mathrm{tri},\overline{\rho},h}\right)^{\mathrm{an}} = X^{\square}_{\mathrm{tri},\overline{\rho},h}$ (resp. $\left(\mathfrak{X}^{\square}_{\mathrm{tri},\overline{\rho},S,h}\right)^{\mathrm{an}} = X^{\square}_{\mathrm{tri},\overline{\rho},S,h}$) and $\left(\mathfrak{X}^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},h}\right)^{\mathrm{an}} = X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},h}$ (resp. $\left(\mathfrak{X}^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},S,h}\right)^{\mathrm{an}} = X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},S,h}$).*

3. Extended eigenvarieties

3.1. **Definitions.** We briefly recall the construction of extended eigenvarieties in the two cases of interest to us. Fix a number field F and a reductive group H over F which is split at all places above p; then we define $\mathbf{G} := \mathrm{Res}_{F/\mathbf{Q}}\,\mathrm{H}$. If we choose split models $\mathrm{H}_{\mathscr{O}_{F_v}}$ over $\mathscr{O}_{F_v}$ for each place $v \mid p$, along with split maximal tori and Borel subgroups $\mathrm{T}_v \subset \mathbf{B}_v \subset \mathrm{H}_{\mathscr{O}_{F_v}}$, we obtain an integral model $\mathbf{G}_{\mathbf{Z}_p} := \prod_{v|p} \mathrm{H}_{\mathscr{O}_{F_v}}$ of $\mathbf{G}$, as well as closed subgroup schemes

$$\mathrm{T} := \prod_{v|p} \mathrm{Res}_{\mathscr{O}_{F_v}/\mathbf{Z}_p} T_v \subset \mathbf{B} := \prod_{v|p} \mathrm{Res}_{\mathscr{O}_{F_v}/\mathbf{Z}_p} \mathbf{B}_v$$

Let $T_0 := \mathrm{T}(\mathbf{Z}_p)$, and let the Iwahori subgroup $I \subset \mathbf{G}_{\mathbf{Z}_p}(\mathbf{Z}_p)$ be the preimage of $\mathbf{B}(\mathbf{F}_p)$ under the reduction map $\mathbf{G}_{\mathbf{Z}_p}(\mathbf{Z}_p) \to \mathbf{G}_{\mathbf{Z}_p}(\mathbf{F}_p)$.

We choose a tame level by choosing compact open subgroups $K_\ell \subset \mathbf{G}(\mathbf{Q}_\ell)$ for each prime $\ell \neq p$, such that $K_\ell = \mathcal{G}(\mathbf{Z}_\ell)$ for almost all primes ℓ (where $\mathcal{G}$ is some reductive model of $\mathbf{G}$ over $\mathbf{Z}[1/M]$ for some integer M). Then we put $K^p := \prod_{\ell \neq p} K_\ell$ and $K := K^p I$; we assume throughout that K contains an open normal subgroup K' such that $[K : K']$ is prime to p and

$$x^{-1} D^\times x \cap K' \subset \mathscr{O}_F^{\times,+} \quad \text{for all } x \in (\mathbf{A}_{F,f} \otimes_F D)^\times \tag{3.1.1}$$

which is the neatness hypothesis of [JN19b].[1] If $\mathbf{Z}$ denotes the center of $\mathbf{G}$, we let $Z(K) := \mathbf{Z}(\mathbf{Q}) \cap K$ and let $\overline{Z(K)} \subset T_0$ denote its p-adic closure. We also let $K_\infty \subset \mathbf{G}(\mathbf{R})$ be a maximal compact and connected subgroup at infinity, and let $Z_\infty^\circ \subset Z_\infty =: \mathbf{Z}(\mathbf{R})$ denote the identity component.

Finally, let $\Sigma \subset T_0$ be the kernel of some splitting of the inclusion $T_0 \subset \mathrm{T}(\mathbf{Q}_p)$; there are then certain submonoids $\Sigma^{\mathrm{cpt}} \subset \Sigma^+ \subset \Sigma$, and we fix some $t \in \Sigma^{\mathrm{cpt}}$.

[1]The authors assume throughout that the level is neat; to relax this assumption, one chooses an open normal subgroup $K' \subset K$ of index prime to p such that K' is neat, and considers the complexes $C_c^\bullet(K', -)^{K/K'}$ and $C_\bullet^{\mathrm{BM}}(K', -)_{K/K'}$. Since K/K' has order prime to p, the finite-slope subcomplexes $C_c^\bullet(K, \mathscr{D}_\kappa)_{\leq h}^{K/K'}$ and $C_\bullet^{\mathrm{BM}}(K', -)_{\leq h, K/K'}$ remain perfect.

In the cases of interest to us, F will be a totally real field, completely split at p, and H will be either GL_2 or the reductive group $\underline{D}^\times$ corresponding to the units of a totally definite quaternion algebra over F split at every place above p. Fixing isomorphisms $D_v \xrightarrow{\sim} \mathrm{Mat}_2(F_v)$ for each place v where D is split, we may define integral models of H_v via $\mathrm{H}_{\mathscr{O}_{F_v}}(R_0) := (R_0 \otimes \mathrm{Mat}_2(\mathscr{O}_{F_v}))^\times$ for all $\mathscr{O}_{F_v}$-algebras R_0 (whether $\mathrm{H} = \mathrm{GL}_2$ or $\underline{D}^\times$). In either case, we let $\mathbf{B}_v \subset \mathrm{H}_{\mathscr{O}_{F_v}}$ be the standard upper-triangular Borel and we let $\mathrm{T}_v \subset \mathbf{B}_v$ be the standard diagonal maximal torus.

For either choice of H, the adelic subgroup $K(N) \subset (\mathbf{A}_{F,f} \otimes \mathrm{H}(F))^\times$ of full level N is neat for $N \geq 3$ such that N is prime to the finite places v where $\mathrm{H}_v \neq \mathrm{GL}_2$. Thus, if we assume $p \geq 5$, we may take K^p arbitrary.

For either choice of H, we define $\Sigma_v^+ := \left\{ \left(\begin{smallmatrix} \varpi_v^{a_1} & 0 \\ 0 & \varpi_v^{a_2} \end{smallmatrix} \right) \mid a_2 \geq a_1 \right\}$ and $\Delta_v := I_v \Sigma_v^+ I_v$. Similarly, we define $\Sigma^+ := \prod_{v|p} \Sigma_v^+$ and $\Delta_p := I\Sigma^+ I = \prod_{v|p} \Delta_v$. Then we fix $U_v := \left[I_v \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix} \right) I_v \right] \in I_v \backslash \mathrm{H}(F_v)/I_v$ and $U_p := \prod_{v|p} U_v$.

For each prime $\ell \neq p$, we fix a monoid $\Delta_\ell \subset \mathbf{G}(\mathbf{Q}_\ell)$ containing K_ℓ, which is equal to $\mathbf{G}(\mathbf{Q}_\ell)$ when $K_\ell = \mathcal{G}(\mathbf{Z}_\ell)$, such that (Δ_ℓ, K_ℓ) is a Hecke pair and the Hecke algebra $\mathbb{T}(\Delta_\ell, K_\ell)$ over $\mathbf{Z}_p$ is commutative. Then we define $\Delta^p := \prod'_{\ell \neq p} \Delta_\ell$ and $\Delta := \Delta^p \Delta_p$. We write $\mathbb{T}(\Delta^p, K^p) := \otimes_{\ell \neq p} \mathbb{T}(\Delta_\ell, K_\ell)$ and $\mathbb{T}(\Delta, K) := \otimes_\ell \mathbb{T}(\Delta_\ell, K_\ell)$ for the corresponding global Hecke algebras.

A *weight* is a continuous homomorphism $\kappa : T_0 \to R^\times$ which is trivial on $Z(K)$, where R is a pseudoaffinoid algebra over $\mathbf{Z}_p$. We define *weight space* $\mathscr{W}$ via

$$\mathscr{W}(R) := \{\kappa \in \mathrm{Hom}_{\mathrm{cts}}(T_0, R^\times) \mid \kappa|_{Z(K)} = 1\}$$

It can be written explicitly as the analytic locus of $\mathrm{Spa}\left(\mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!], \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]\right)$. Then $\mathscr{W}$ is equidimensional of dimension $1 + [F : \mathbf{Q}] + \mathfrak{d}$, where $\mathfrak{d}$ is the defect in Leopoldt's conjecture for F and p.

The next step is to construct a sheaf of Hecke modules over weight space, such that U_p acts compactly and admits a Fredholm determinant. We will actually use two such sheaves. If $\kappa : T_0 \to R^\times$ is a weight, then [JN16] construct certain modules of analytic functions $\mathcal{A}_\kappa^r$ and distributions $\mathcal{D}_\kappa^r$. Here $r \in (r_\kappa, 1)$, where $r_\kappa \in [1/p, 1)$. When $r_\kappa \in (1/p, 1)$, they also construct $\mathcal{A}_\kappa^{<r}$ and $\mathcal{D}_\kappa^{<r}$, so that $\mathcal{D}_\kappa^r$ is the dual of $\mathcal{A}_\kappa^{<r}$ and $\mathcal{A}_\kappa^r$ is the dual of $\mathcal{D}_\kappa^{<r}$. As in [HN17] we fix augmented Borel–Serre complexes $C_\bullet^{\mathrm{BM}}(K, -)$ and $C_c^\bullet(K, -)$ for Borel–Moore homology and compactly supported cohomology, respectively, and we consider

$$C_\bullet^{\mathrm{BM}}(K, \mathcal{A}_\kappa^r)$$

as well as

$$C_c^\bullet(K, \mathcal{D}_\kappa^r) \quad \text{and} \quad C_c^\bullet(K, \mathcal{D}_\kappa^{<r})$$

Now $\mathcal{A}_\kappa^r$ and $\mathcal{D}_\kappa^r$ are potentially orthonormalizable, so $C_*^{\mathrm{BM}}(K, \mathcal{A}_\kappa^r) := \oplus_i C_i^{\mathrm{BM}}(K, \mathcal{A}_\kappa^r)$ and $C_c^*(K, \mathcal{D}_\kappa^r) := \oplus_i C_c^i(K, \mathcal{D}_\kappa^r)$ are, as well. Since U_p acts compactly on $\mathcal{A}_\kappa^r$

and $\mathcal{D}^r_\kappa$, this implies that there are Fredholm determinants $F^{r,\prime}_\kappa$ and F^r_κ for its action on $C^{\mathrm{BM}}_*(K, \mathcal{A}^r_\kappa)$ and $C^*_c(K, \mathcal{D}^r_\kappa)$, respectively.

It turns out that $F^{r,\prime}_\kappa$ and F^r_κ are independent of r, by [JN16, Proposition 4.1.2]; we set $\mathscr{D}_\kappa := \varprojlim_r \mathcal{D}^r_\kappa$ and $\mathscr{A}_\kappa := \varinjlim_r \mathcal{A}^r_\kappa$, and we write F_κ and F'_κ for the Fredholm determinants of U_p on $C^*_c(K, \mathscr{D}_\kappa)$ and $C^{\mathrm{BM}}_*(K, \mathscr{A}_\kappa)$, respectively. Then F_κ and F'_κ define spectral varieties $\mathscr{Z} \subset \mathbf{A}^1_{\mathscr{W}_F}$ and $\mathscr{Z}' \subset \mathbf{A}^1_{\mathscr{W}_F}$. We let $\pi : \mathscr{Z} \to \mathscr{W}_F$ and $\pi' : \mathscr{Z}' \to \mathscr{W}_F$ be the projection on the first factor.

By [JN16, Theorem 2.3.2], $\mathscr{Z}$ has a cover by open affinoid subspaces V such that $U := \pi(V)$ is an open affinoid subspace of $\mathscr{W}_F$ and $\pi|_V : V \to U$ is finite of constant degree. This implies that over such a V, F factors as $F_V = Q_V S_V$ where Q_V is a multiplicative polynomial of degree $\deg \pi|_V$, S_V is a Fredholm series, and Q_V and S_V are relatively prime.

If such a factorization exists, we may make $C^\bullet_c(K, \mathscr{D}_V)$ into a complex of $\mathscr{O}_{\mathscr{Z}}$-modules by letting T act via U_p^{-1}. Then the assignment $V \mapsto \ker Q^*_V(U_p) \subset C^\bullet(K, \mathscr{D}_V)$ defines a bounded complex $\mathscr{K}^\bullet$ of coherent $\mathscr{O}_{\mathscr{Z}}$-modules, where $Q^*_V(T) := T^{\deg Q_V} Q_V(1/T)$. If $V = \pi^{-1}(U)$, where (U, h) is a slope datum, then $\mathscr{K}^\bullet$ is the slope-$\leq h$ subcomplex of $C^\bullet_c(K, \mathscr{D}_V)$. We set

$$\mathscr{M}^*_c := \oplus_i H^i(\mathscr{K})$$

which is a coherent sheaf on $\mathscr{Z}$.

Such factorizations exist locally, by an extension of a result of [AS]:

Proposition 3.1.1. *Let R be a pseudoaffinoid algebra, and let $x_0 \in \operatorname{Spa} R$ be a maximal point. Let $F(T) \in R\{\{T\}\}$ be a Fredholm power series and fix $h \in \mathbf{Q}$. Suppose $F_{x_0} \neq 0$, and let $F_{x_0} = Q_0 S_0$ be the slope $\leq h$-factorization of the specialization of F at x_0. Then there is an open affinoid subspace $U \subset \operatorname{Spa} R$ containing x_0 such that F_U has a slope $\leq h$-factorization $F_U = QS$ with Q specializing to Q_0 and S specializing to S_0 at x_0.*

Proof. The existence of the factorization of F_{x_0} follows from the version of the Weierstrass preparation theorem proved in [AS, Lemma 4.4.3]. Then the proof of the proposition is nearly identical to that of [AS, Theorem 4.5.1], up to replacing p with u and translating the numerical inequalities into rational localization conditions. $\square$

We further observe that we have inclusions $\mathcal{D}^r_\kappa \subset \mathcal{D}^{<r}_\kappa \subset \mathcal{D}^s_\kappa$ for any $r_\kappa \leq s < r$. Thus, the fact that $F^r_\kappa = F^s_\kappa$ implies that $\mathscr{M}^*_c = \oplus_i H^i_c(K, \mathscr{D}^{<r}_\kappa)_{\leq h}$ for any $r > r_\kappa$.

We may carry out the same procedure for the action of U_p on $C^{\mathrm{BM}}_*(K, \mathscr{A}_\kappa)$, and obtain a coherent sheaf $\mathscr{M}^{\mathrm{BM}}_* = \oplus_i H^{\mathrm{BM}}_i(K, \mathscr{A}_\kappa)_{\leq h}$ on $\mathscr{Z}'$. Let $\mathbb{T}$ denote either $\mathbb{T}(\Delta^p, K^p)$ or $\mathbb{T}(\Delta, K)$. Both $\mathscr{M}^*_c$ and $\mathscr{M}^{\mathrm{BM}}_*$ are Hecke modules, so we have constructed *eigenvariety data* $(\mathscr{Z}, \mathscr{M}^*_c, \mathbb{T}, \psi)$ and $(\mathscr{Z}', \mathscr{M}^{\mathrm{BM}}_*, \mathbb{T}, \psi')$

(where $\psi : \mathbb{T} \to \mathrm{End}_{\mathscr{O}_{\mathscr{X}}}(\mathscr{M}_c^*)$ and $\psi' : \mathbb{T} \to \mathrm{End}_{\mathscr{O}_{\mathscr{X}'}}(\mathscr{M}_*^{\mathrm{BM}})$ give the Hecke-module structures).

Finally, we may construct eigenvarieties from the eigenvariety data. Let $\mathscr{T}$ and $\mathscr{T}'$ denote the sheaves of $\mathscr{O}_{\mathscr{X}}$-algebras generated by the images of ψ and ψ', respectively; in particular, if $\mathscr{X}_{U,h} \subset \mathscr{X}$ is an open affinoid corresponding to the slope datum (U, h), then

$$\mathscr{T}(\mathscr{X}_{U,h}) = \mathrm{im}\left(\mathscr{O}(\mathscr{X}_{U,h}) \otimes_{\mathbf{Z}_p} \mathbb{T} \to \mathrm{End}_{\mathscr{O}(\mathscr{X}_{U,h})}\left(H_c^*(K, \mathscr{D}_U)_{\leq h}\right)\right) =: \mathbb{T}_{U,h}$$

and

$$\mathscr{T}'(\mathscr{X}'_{U,h}) = \mathrm{im}\left(\mathscr{O}(\mathscr{X}'_{U,h}) \otimes_{\mathbf{Z}_p} \mathbb{T} \to \mathrm{End}_{\mathscr{O}(\mathscr{X}'_{U,h})}\left(H_*^{\mathrm{BM}}(K, \mathscr{A}_U)_{\leq h}\right)\right) =: \mathbb{T}'_{U,h}$$

Then we set

$$\mathscr{X}_{\mathbf{G}}^{\mathbb{T}} := \underline{\mathrm{Spa}}\mathscr{T}$$

and

$$\mathscr{X}_{\mathbf{G}}^{\mathbb{T},\prime} := \underline{\mathrm{Spa}}\mathscr{T}'$$

and we have finite morphisms $q : \mathscr{X}_{\mathbf{G}} \to \mathscr{X}$ and $q' : \mathscr{X}'_{\mathbf{G}} \to \mathscr{X}'$, and $\mathbf{Z}_p$-algebra homomorphisms $\phi_{\mathscr{X}} : \mathbb{T} \to \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathbb{T}})$ and $\phi_{\mathscr{X}'} : \mathbb{T} \to \mathscr{O}(\mathscr{X}_{\mathbf{G}}^{\mathbb{T},\prime})$. If the choice of Hecke operators is clear from context, we will drop $\mathbb{T}$ from the notation.

If $\mathbb{T} = \mathbb{T}(\Delta, K)$, then unlike [JN16], we are adding the Hecke operators U_v at places $v \mid p$ to our Hecke algebras (and hence to the coordinate rings of our eigenvarieties), not just the controlling operator U_p.

3.2. **The middle-degree eigenvariety.** When $F = \mathbf{Q}$ and $\mathbf{G} = \mathrm{H} = \mathrm{GL}_2$, for any fixed slope h such that $C_c^\bullet(K, \mathscr{D}_\kappa)$ has a slope-$\leq h$ decomposition, the complex $C_c^\bullet(K, \mathscr{D}_\kappa)_{\leq h}$ has cohomology only in degree 1, and $H_c^1(K, \mathscr{D}_\kappa)_{\leq h}$ is projective. As a result, the eigencurve is reduced and equidimensional, and classical points are very Zariski-dense. For a general totally real field F, the situation is more complicated. The complex $C_c^\bullet(K, \mathscr{D}_\kappa)_{\leq h}$ lives in degrees $[0, 2d]$ and we are still primarily interested in the degree-d cohomology; indeed, the discussion of [Har87, §3.6] shows that cuspidal cohomological automorphic forms contribute only to middle degree cohomology in the classical finite-dimensional classical analogue. However, there is no reason to expect the other cohomology groups to vanish.

Instead, following [BH17] we will sketch the construction of an open subspace $\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}} \subset \mathscr{X}_{\mathrm{GL}_2/F}$ where $H_c^i(K, \mathscr{D}_\kappa)$ vanishes for $i \neq d$; by [BH17, Theorem B.0.1], all classical points of $\mathscr{X}_{\mathrm{GL}_2/F}$ whose associated Galois representation have sufficiently large residual image lie in $\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}}$. The cohomology and base change result [JN16, Theorem 4.2.1] shows that the locus where $H_c^i(K, \mathscr{D}_\kappa) = 0$ for $i \geq d+1$ is open, but we need to use the homology complexes $C_\bullet^{\mathrm{BM}}(K, \mathcal{A}_\kappa)$ to control $H_c^i(K, \mathscr{D}_\kappa)$ for $i \leq d-1$.

As in [BH17], the key points are a base change result for Borel–Moore homology, and a universal coefficients theorem relating it to compactly supported cohomology:

Proposition 3.2.1. • *There is a third-quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Tor}_{-i}^{R}(H_{-j}^{\mathrm{BM}}(K, \mathscr{A}_\kappa)_{\leq h}, S) \Rightarrow H_{-i-j}^{\mathrm{BM}}(K, \mathscr{A}_{\kappa_S})_{\leq h}$$

• *There is a second-quadrant spectral sequence*

$$E_2^{i,j} = \mathrm{Ext}_R^i(H_j^{\mathrm{BM}}(K, \mathscr{A}_\kappa)_{\leq h}, R) \Rightarrow H_c^{i+j}(K, \mathscr{D}_\kappa)_{\leq h}$$

These are spectral sequences of $\mathbb{T}(\Delta, K)$*-modules.*

The proof uses both the fact that $\mathcal{D}_\kappa^{<r}$ is the continuous dual of $\mathcal{A}_\kappa^r$, and the fact that $H_c^i(K, \mathcal{D}_\kappa^{<r})_{\leq h} = H_c^i(K, \mathcal{D}_\kappa^r)_{\leq h}$ for all $r > r_\kappa$.

Proposition 3.2.2. *If* (U, h) *is a slope datum, then we have a natural commuting diagram*

$$\begin{tikzcd} \mathscr{O}(U) \otimes \mathbb{T}(\Delta, K) \arrow[r] \arrow[d] & \mathbb{T}'_{U,h} \arrow[d, dashed] \\ \mathbb{T}'_{U,h} \arrow[r] & \mathbb{T}^{\mathrm{red}}_{U,h} \end{tikzcd}$$

Thus, we have a morphism $\tau : \mathscr{X}^{\mathrm{red}}_{\mathrm{GL}_2/F} \to \mathscr{X}'_{\mathrm{GL}_2/F}$ and a closed immersion $i : \mathscr{X}^{\mathrm{red}}_{\mathrm{GL}_2/F} \hookrightarrow \mathscr{X}_{\mathrm{GL}_2/F}$.

Definition 3.2.3.

$$\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}} := \mathscr{X}_{\mathrm{GL}_2/F} \smallsetminus \left[\left(\cup_{j=d+1}^{2d} \mathrm{supp}(\mathscr{M}_c^j)\right) \cup \left(\cup_{j=0}^{d-1} \mathrm{supp}(i_*\tau^*\mathscr{M}_j^{\mathrm{BM}}\right)\right]$$

By construction, a point $x \in \mathscr{X}_{\mathrm{GL}_2/F}$ of weight λ_x lies in the Zariski-open subspace $\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}} \subset \mathscr{X}_{\mathrm{GL}_2/F}$ if and only if $H_c^j(K, k_x \otimes \mathscr{D}_{\lambda_x})_{\mathfrak{m}_x} = 0$ for all $j \neq d$ (where $\mathfrak{m}_x$ is the maximal ideal of the Hecke algebra corresponding to x).

Proposition 3.2.4. (1) *The coherent sheaf* $\mathscr{M}_c^d|_{\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}}}$ *is flat over* $\mathscr{W}$.

(2) $\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}}$ *is covered by open affinoids* W *such that* W *is a connected component of* $(\pi \circ q)^{-1}(U)$*, where* (U, h) *is some slope datum, and* $\mathscr{T}(W)$ *acts faithfully on* $\mathscr{M}_c^d(W) \cong e_W H_c^d(K, \mathscr{D}_\kappa)_{\leq h}$ *(where* e_W *is the idempotent projector restricting from* $(\pi \circ q)^{-1}(U)$ *to* W*).*

Proof. This follows from the base change spectral sequence, and the criterion for flatness. □

3.3. **Jacquet–Langlands.** The classical Jacquet–Langlands correspondence lets us transfer automorphic forms between GL_2 and quaternionic algebraic groups. Over $\mathbf{Q}$, this correspondence was interpolated in [Che05] to give a closed immersion of eigencurves $\mathscr{X}^{\mathrm{rig}}_{D^\times/\mathbf{Q}} \hookrightarrow \mathscr{X}^{\mathrm{rig}}_{\mathrm{GL}_2/\mathbf{Q}}$; this interpolation was given for general totally real fields in [Bir19]. We give the corresponding result for extended eigenvarieties. However, as we have elected to work with the eigenvariety for GL_2/F constructed in [JN16] via overconvergent cohomology, instead of the eigenvariety constructed from Hilbert modular forms, we will never get an isomorphism of eigenvarieties, even when $[F:\mathbf{Q}]$ is even.

Let D be a totally definite quaternion algebra over F, split at every place above p, and let $\mathfrak{d}_D$ be its discriminant. For any ideal $\mathfrak{n} \subset \mathscr{O}_F$ with $(\mathfrak{d}_D, \mathfrak{n}) = 1$, we define the subgroup $K_1^{D^\times}(\mathfrak{n}) \subset (\mathscr{O}_D \otimes \widehat{\mathbf{Z}})^\times$

$$K_1^{D^\times}(\mathfrak{n}) := \left\{ g \in (\mathscr{O}_D \otimes \widehat{\mathbf{Z}})^\times \mid g \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{\mathfrak{n}} \right\}$$

We may define a similar subgroup $K_1^{\mathrm{GL}_2/F}(\mathfrak{n}) \subset \mathrm{Res}_{\mathscr{O}_F/\mathbf{Z}_p} \mathrm{GL}_2(\widehat{\mathbf{Z}})$.

A *classical algebraic weight* is a tuple $(k_\sigma) \in \mathbf{Z}_{\geq 2}^{\Sigma_\infty}$ together with a tuple $(v_\sigma) \in \mathbf{Z}^{\Sigma_\infty}$ such that $(k_\sigma) + (v_\sigma) = (r, \dots, r)$ for some $r \in \mathbf{Z}$, where Σ_∞ is the set of embeddings $F \hookrightarrow \mathbf{R}$. Set $e_1 := (\frac{r+k_\sigma}{2})$ and $e_2 := (\frac{r-k_\sigma}{2})$, and define characters $\kappa_i : F^\times \to \mathbf{R}^\times$ for $i = 1, 2$ via

$$\kappa_i(x) = \prod_{\sigma \in \Sigma_\infty} \sigma(x)^{e_{i,\sigma}}$$

Then (κ_1, κ_2) is a character on $\mathrm{T}(\mathbf{Z})$ which is trivial on a finite-index subgroup of the center $Z_G(\mathbf{Z}) = \mathscr{O}_F^\times$.

Then we have the classical Jacquet–Langlands correspondence:

Theorem 3.3.1. *Let κ be a classical weight, and let $\mathfrak{n} \subset \mathscr{O}_F$ be an ideal such that $(\mathfrak{n}, \mathfrak{d}_D) = 1$. There is a Hecke-equivariant isomorphism of spaces of cusp forms*

$$S_\kappa^{D^\times}(K_1^{D^\times}(\mathfrak{n})) \xrightarrow{\sim} S_\kappa^{\mathfrak{d}_D-\mathrm{new}}(K_1^{\mathrm{GL}_2/F}(\mathfrak{n}\mathfrak{d}_D))$$

We will interpolate this correspondence to a closed immersion $\mathscr{X}_{\underline{D}^\times} \hookrightarrow \mathscr{X}_{\mathrm{GL}_2/F}$, where the source has tame level $K_1^{D^\times}(\mathfrak{n})$ and the target has tame level $K_1^{\mathrm{GL}_2/F}(\mathfrak{n})$. We use the interpolation theorem of [JN19a]:

Theorem 3.3.2 ([JN19a, Theorem 3.2.1])**.** *Let $\mathfrak{D}_i = (\mathscr{Z}_i, \mathscr{M}_i, \mathbb{T}_i, \psi_i)$ for $i = 1, 2$ be eigenvariety data, with corresponding eigenvarieties $\mathscr{X}_i$, and suppose we have the following:*

- *A morphism $j : \mathscr{Z}_1 \to \mathscr{Z}_2$*
- *A $\mathbf{Z}_p$-algebra homomorphism $\mathbb{T}_2 \to \mathbb{T}_1$*
- *A subset $\mathscr{X}^{\mathrm{cl}} \subset \mathscr{X}_1$ of maximal points such that the $\mathbb{T}_2$-eigensystem of x appears in $\mathscr{M}_2(j(\mathfrak{q}_1(x)))$ for all $x \in \mathscr{X}^{\mathrm{cl}}$.*

Let $\overline{\mathscr{X}} \subset \mathscr{X}_1$ denote the Zariski closure of $\mathscr{X}^{\mathrm{cl}}$ (with its underlying reduced structure). Then there is a canonical morphism $i : \overline{\mathscr{X}} \to \mathscr{X}_2$ lying over j, such that $\phi_{\overline{\mathscr{X}}} \circ \sigma = i^* \circ \phi_{\mathscr{X}_2}$. If j is a closed immersion and σ is a surjection, then i is a closed immersion.

We take $\mathscr{Z}_1 = \mathscr{Z}_2 = \mathscr{W}_F \times \mathbf{G}_m$. In order to define $\mathbb{T} = \mathbb{T}_1 = \mathbb{T}_2$, we set

$$\Delta_v = \begin{cases} \mathrm{GL}_2(F_v) & \text{if } v \nmid p\mathfrak{d}_D\mathfrak{n} \\ K_1^{D^\times}(\mathfrak{n})_v & \text{if } v \mid \mathfrak{d}_D\mathfrak{n} \end{cases}$$

For $v \mid p$, we take Δ_v as in §3.1. In other words, $\mathbb{T}$ is the commutative $\mathbf{Z}_p$-algebra generated by $T_v := [K_v \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right) K_v]$ and $S_v := [K_v \left(\begin{smallmatrix} \varpi_v & \\ & \varpi_v \end{smallmatrix}\right) K_v]$ for $v \nmid p\mathfrak{d}_D\mathfrak{n}$ and U_v for $v \mid p$.

However, we cannot immediately combine this interpolation theorem with the Jacquet–Langlands correspondence, because our choice of weight space means that classical weights may not be Zariski dense unless Leopoldt's conjecture is true. More precisely, given a classical algebraic weight, we constructed a character on $\mathrm{T}(\mathbf{Z})$ trivial on a finite-index subgroup of $\mathscr{O}_F^\times$, and conversely, characters on $\mathrm{T}(\mathbf{Z})$ trivial on a finite-index subgroup of $\mathscr{O}_F^\times$ yield classical algebraic weights. This equivalence relies on Dirichlet's unit theorem.

This means that there are two natural definitions of p-adic families of weights, $\mathscr{W}_F' = \mathrm{Spa}\, \mathbf{Z}_p[\![(\mathrm{Res}_{\mathscr{O}_F/\mathbf{Z}}\, \mathbf{G}_m) \times \mathbf{Z}_p^\times]\!]^{\mathrm{an}}$ interpolating classical algebraic weights, and $\mathscr{W}_F$ interpolating characters on T_0, and the equivalence of those two definitions depends on Leopoldt's conjecture.

Fortunately, the gap between these weight spaces can be controlled: there is a closed embedding $\mathscr{W}_F' \hookrightarrow \mathscr{W}_F$, and the twisting action by characters on $\mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}}$ defines a surjective map

$$\widehat{\mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}}} \times \mathscr{W}_F' \to \mathscr{W}_F^{\mathrm{rig}}$$

We say that a weight $\lambda \in \mathscr{W}_F^{\mathrm{rig}}(\overline{\mathbf{Q}}_p)$ is *twist classical* if it is in the $\widehat{\mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}}}(\overline{\mathbf{Q}}_p)$-orbit of a classical weight. Then twist classical weights are very Zariski dense in $\mathscr{W}_F$.

In addition, we may define a twisting action on Hecke modules, as in [BH17]. Let $\mathrm{Gal}_{F,p}$ denote the Galois group of the maximal abelian extension of F unramified away from p and ∞, and let $\eta : \mathrm{Gal}_{F,p} \to \overline{\mathbf{Q}}_p^\times$ be a continuous character. Global class field theory implies that $\mathrm{Gal}_{F,p}$ fits into an exact sequence

$$1 \to \mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}} \to \mathrm{Gal}_{F,p} \to \mathrm{Cl}_F^+ \to 1$$

where Cl_F^+ is the narrow class group of F (and hence finite). Suppose M is an R-module equipped with an R-linear left Δ_p-action. Then we may define a new left Δ_p-module $M(\eta) := M \otimes \eta^{-1}|_{\mathscr{O}_{F,p}^\times}$, where the action of $g \in \Delta_p$ is given by

$$g \cdot m = \left(\eta^{-1}|_{\mathscr{O}_{F,p}^\times}(\det g \cdot p^{-\sum_{v|p} v(\det g)})\right) \cdot (g \cdot m)$$

In particular, $\mathscr{D}_\kappa(\eta) \cong \mathscr{D}_{\eta^{-1}\cdot\kappa}$ by [BH17, Lemma 5.5.2], and there is an isomorphism

$$\mathrm{tw}_\eta : H_c^*(K, \mathscr{D}_\kappa) \xrightarrow{\sim} H_c^*(K, \mathscr{D}_{\eta^{-1}\cdot\kappa}$$

Suppose $x \in \mathscr{X}_{\underline{D}^\times}(\overline{\mathbf{Q}}_p)$ is a point with $\mathrm{wt}(x) =: \lambda$, corresponding to the system of Hecke eigenvalues $\psi_x : \mathbb{T} \to \overline{\mathbf{Q}}_p$. Then we define a new system of Hecke eigenvalues, via

$$\mathrm{tw}_\eta(\psi_x)(T) = \begin{cases} \eta(\varpi_v)\psi_x(T) & \text{if } v \nmid p\mathfrak{d}_D\mathfrak{n} \text{ and } T = T_v \\ \eta(\varpi_v)^2\psi_x(T) & \text{if } v \nmid p\mathfrak{d}_D\mathfrak{n} \text{ and } T = S_v \\ \eta(\varpi_v)\psi_x(T) & \text{if } v \mid p \end{cases}$$

Then it follows from [BH17, Proposition 5.5.5] that $\mathrm{tw}_\eta(\psi_x)$ corresponds to a point $\mathrm{tw}_\eta(x) \in \mathscr{X}_{\underline{D}^\times}$ of weight $\eta^{-1}|_{\mathscr{O}_{F,p}^\times} \cdot \kappa$.

We say that a point $x \in \mathscr{X}_{\underline{D}^\times}(\overline{\mathbf{Q}}_p)$ is *twist classical* if it is in the $\widehat{\mathrm{Gal}_{F,p}}(\overline{\mathbf{Q}}_p)$-orbit of a point corresponding to a classical system of Hecke eigenvalues.

Proposition 3.3.3. *Twist classical points are very Zariski dense in $\mathscr{X}_{\underline{D}^\times}$.*

Proof. Recall that $\mathscr{X}_{\underline{D}^\times}$ admits a cover by affinoid pseudorigid spaces of the form $\mathrm{Spa}\,\mathscr{T}(\mathscr{Z}_{U,h})$, where $\pi : \mathscr{Z}_{U,h} \to U$ is finite of constant degree, and

$$\mathscr{T}(\mathscr{Z}_{U,h}) = \mathrm{im}\left(\mathscr{O}(\mathscr{Z}_{U,h}) \otimes_{\mathbf{Z}_p} \mathbb{T}^p \to \mathrm{End}_{\mathscr{O}(\mathscr{Z}_{U,h})}(H_c^*(K, \mathscr{D}_U)_{\leq h}\right)$$

We write $U = \mathrm{Spa}\,R$ for some pseudo-affinoid algebra R over $\mathbf{Z}_p$. We will show that $\mathrm{Spec}\,\mathscr{T}(\mathscr{Z}_{U,h}) \to \mathrm{Spec}\,R$ carries irreducible components surjectively onto irreducible components, and we will construct a Zariski dense set of maximal points $W_{U,h}^{\mathrm{tw-cl}} \subset U$ such that the points of $\mathrm{wt}^{-1}(W_{U,h}^{\mathrm{tw-cl}})$ are twist classical. By [Che04, Lemme 6.2.8], this implies the desired result.

To see that irreducible components of $\mathrm{Spec}\,\mathscr{T}(\mathscr{Z}_{U,h})$ map surjectively onto irreducible components of $\mathrm{Spec}\,R$, we observe that D is totally definite, so the associated Shimura manifold is a finite set of points and $H_c^*(K, \mathscr{D}_U)$ vanishes outside degree 0. The base change spectral sequence of [JN16, Theorem 4.2.1] implies that the formation of $H^0(K, \mathscr{D}_U)_{\leq h}$ commutes with arbitrary base change on U, which implies that $H^0(K, \mathscr{D}_U)_{\leq h}$ is flat. Then [Che04, Lemme 6.2.10] implies that $\mathrm{Spec}\,\mathscr{T}(\mathscr{Z}_{U,h}) \to \mathrm{Spec}\,R$ carries irreducible components surjectively onto irreducible components, as desired.

Thus, it remains to construct $W_{U,h}^{\mathrm{tw-cl}}$. Birkbeck proved a "small slope implies classical" result [Bir19, Theorem 4.3.7], and constructed a set $W_{U,h}^{\mathrm{cl}}$ Zariski dense in $U \cap \mathscr{W}_F'$ such that the points of $\mathrm{wt}^{-1}(W_{U,h}^{\mathrm{cl}})$ are classical (see the proof of [Bir19, Theorem 6.1.9]). Setting $W_{U,h}^{\mathrm{tw-cl}}$ to be the $\widehat{\mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}}}(\overline{\mathbf{Q}}_p)$-orbit of $W_{U,h}^{\mathrm{cl}}$, [BH17, Lemma 6.3.1] implies that points of $\mathrm{wt}^{-1}(W_{U,h}^{\mathrm{tw-cl}})$ are twist classical, and we are done. □

As a corollary, we deduce that $\mathscr{X}_{\underline{D}^\times}$ has no components supported entirely in characteristic p:

Corollary 3.3.4. *$\mathscr{X}_{\underline{D}^\times}^{\mathrm{rig}}$ is Zariski dense in $\mathscr{X}_{\underline{D}^\times}$.*

We may use similar arguments to show that $\mathscr{X}_{\underline{D}^\times}$ is reduced:

Proposition 3.3.5. *The eigenvariety $\mathscr{X}_{\underline{D}^\times/F}$ is reduced.*

Proof. We first show that $\mathscr{X}_{\underline{D}^\times}^{\mathrm{rig}}$ is reduced. By [JN16, Proposition 6.1.2] (which adapts [Che05, Proposition 3.9]), it is enough to find a Zariski dense set of twist classical weights $W_{U,h}^{\mathrm{ss}} \subset U \subset \mathscr{W}_F^{\mathrm{rig}}$ for each slope datum (U,h) such that $\mathscr{M}(\mathscr{Z}_{U,h})_\kappa$ is a semi-simple Hecke module for all $\kappa \in W_{U,h}^{\mathrm{ss}}$. Birkbeck [Bir19, Lemma 6.1.12] constructed sets $W_{U,h}^{\prime,\mathrm{ss}}$ Zariski dense in $U \cap \mathscr{W}_F^{\prime,\mathrm{rig}}$ with this property, and we will again use twisting by p-adic characters to construct $W_{U,h}^{\mathrm{ss}}$.

If $\eta : \mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}} \to \overline{\mathbf{Q}}_p^\times$ is a character, we have an isomorphism

$$\mathrm{tw}_\eta : H_c^*(K, \mathscr{D}_\kappa) \xrightarrow{\sim} H_c^*(K, \mathscr{D}_{\eta^{-1}\cdot\kappa}$$

By [BH17, Proposition 5.5.5], tw_η is Hecke-equivariant up to scalars, so $\mathscr{M}(\mathscr{Z}_{U,h})_\kappa$ is a semi-simple Hecke module if and only if $\mathscr{M}(\mathscr{Z}_{\eta^{-1}\cdot U,h})_{\eta^{-1}\cdot\kappa}$ is. Thus, we may take $W_{U,h}^{\mathrm{ss}}$ to be the $\widehat{\mathscr{O}_{F,p}^\times/\overline{\mathscr{O}_F^{\times,+}}}(\overline{\mathbf{Q}}_p)$-orbit of $\cup_{U'} W_{U',h}^{\prime,\mathrm{ss}}$, as (U',h) varies through slope data, and we see that $\mathscr{X}_{\underline{D}^\times}^{\mathrm{rig}}$ is reduced.

Now let $X \subset \mathscr{X}_{\underline{D}^\times}$ be an open affinoid subspace, and let $\{X_i\}$ be an open affinoid cover of the rigid analytic locus $X^{\mathrm{rig}} \subset X$. Since $X \smallsetminus X^{\mathrm{rig}}$ contains no open subset of X, the natural map

$$\mathscr{O}(X) \to \prod_i \mathscr{O}(X_i)$$

is injective. Each $\mathscr{O}(X_i)$ is reduced, so $\mathscr{O}(X)$ is, as well. □

Now the Jacquet–Langlands correspondence for eigenvarieties follows immediately:

Corollary 3.3.6. *There is a closed immersion $\mathscr{X}_{\underline{D}^\times} \hookrightarrow \mathscr{X}_{\mathrm{GL}_2/F}$ interpolating the classical Jacquet–Langlands correspondence on (twist) classical points, where the source has tame level $K_1^{\underline{D}^\times}(\mathfrak{n})$ and the target has tame level $K_1^{\mathrm{GL}_2/F}(\mathfrak{n})$.*

In particular, if $[F:\mathbf{Q}]$ is even, we can find D split at all finite places and ramified at all infinite places. Then we may take in particular $\mathfrak{n} = \mathscr{O}_F$ to obtain a morphism of eigenvarieties of tame level 1.

3.4. **Cyclic base change.** Fix an integer $N \in \mathbf{N}$, and let S be a finite set of primes containing every prime dividing pN. For any number field F, we again let $K_F^p \subset \mathrm{GL}_2(\mathbf{A}_F)$ be the compact open subgroup given by

$$K_F^p := \{g \in \mathrm{GL}_2(\mathbf{A}_F) \mid g \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{N}\}$$

and we let $K_F := K_F^p I$. We also define the Hecke algebra

$$\mathbb{T}_F^S := \mathbb{T}_{\mathrm{GL}_2/F}^S := \otimes_{v \notin S} \mathbb{T}(\mathrm{GL}_2(F_v), \mathrm{GL}_2(\mathscr{O}_{F_v}))$$

There is a homomorphism $\sigma_F^S : \mathbb{T}_F^S \to \mathbb{T}_{\mathbf{Q}}^S$ induced by unramified local Langlands and restriction of Weil representations from W_F to $W_{\mathbf{Q}}$.

Similarly, there is a morphism of weight spaces $\mathscr{W}_{\mathbf{Q},0} \hookrightarrow \mathscr{W}_{\mathbf{Q}} \to \mathscr{W}_F$ induced by the norm map $T_{F,0} \to T_{\mathbf{Q},0}$.

In the special case where $F/\mathbf{Q}$ is cyclic, [JN19a] interpolated the classical base change map:

Theorem 3.4.1 ([JN19a, Theorem 4.3.1])**.** [2] *There is a finite morphism*

$$\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q},\mathrm{cusp},F-\mathrm{ncm}}^S \to \mathscr{X}_{\mathrm{GL}_2/F}^S$$

lying over $\mathscr{W}_{\mathbf{Q}} \to \mathscr{W}_F$ *and compatible with the homomorphism* σ_F^S.

Here the source includes only cuspidal components with a Zariski-dense set of forms without CM by an imaginary quadratic subfield of F.

We wish to characterize the image of this map when F is totally real and completely split at p (so that the "F - ncm" condition is vacuous). We further assume that $[F:\mathbf{Q}]$ is prime to p.

Remark 3.4.2. We expect that it is possible to construct a base change morphism and characterize its image for more general cyclic extensions of number fields F'/F; however, for simplicity (and compatibility with [JN19a]) we have chosen to restrict to this setting.

Let $\mathrm{Gal}(F/\mathbf{Q}) = \langle \tau \rangle$. Then $\mathrm{Gal}(F/\mathbf{Q})$ acts on $\mathrm{GL}_{2/F}$, stabilizing $\mathrm{T} \subset \mathbf{B}$ and I, and also stabilizing the tame level K_F^p. We will construct a "$\mathrm{Gal}(F/\mathbf{Q})$-fixed $\mathrm{GL}_{2/F}$-eigenvariety" $\mathscr{X}_{\mathrm{GL}_2/F}^{S,\mathrm{Gal}(F/\mathbf{Q})}$ and show that it is the image of the cyclic base change map; Xiang [Xia18] used a similar idea to construct p-adic families of essentially self-dual automorphic representations.

[2] The authors only construct the morphism when $N \geq 5$, to maintain their running assumption that the level is actually neat (as opposed to containing an open neat subgroup with index prime to p). However, the argument is identical for small N.

We first observe that $\mathrm{Gal}(F/\mathbf{Q})$ acts on $\mathbb{T}_F^S$ via $(\tau \cdot T)(g) = T(\tau^{-1}(g))$ for all $T \in \mathbb{T}_F^S$ and $g \in \mathrm{GL}_2(\mathbf{A}_{F,f})$. Then for any $\delta \in \Delta$, $(\tau \cdot [K_F \delta K_F])(g) = [K_F \tau^{-1}(\delta) K_F](g)$, and in particular, $\tau \cdot U_v = U_{\tau(v)}$, and hence $\mathrm{Gal}(F/\mathbf{Q})$ fixes U_p. Similarly, we have an action of $\mathrm{Gal}(F/\mathbf{Q})$ on $\mathscr{W}_{\mathbf{Q}}$ given via $(\tau \cdot \lambda)(g) = \lambda(\tau^{-1}(g))$; the image of $\mathscr{W}_{\mathbf{Q}}$ in $\mathscr{W}_F$ is the diagonal locus, i.e., exactly the $\mathrm{Gal}(F/\mathbf{Q})$-fixed locus.

Since U_p is fixed by $\mathrm{Gal}(F/\mathbf{Q})$, we see that if κ is a weight fixed by $\mathrm{Gal}(F/\mathbf{Q})$, then the Fredholm determinant $F_\kappa(T)$ of the action of U_p on $C^\bullet(K_F, \mathscr{D}_\kappa)$ is fixed by $\mathrm{Gal}(F/\mathbf{Q})$. Thus, we have a spectral variety $\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})} \subset \mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})} \times \mathbf{A}^{1,\mathrm{an}}$ over $\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}$.

Lemma 3.4.3. *Let $\kappa : T_0 \to R^\times$ be a weight fixed by* $\mathrm{Gal}(F/\mathbf{Q})$. *There is an action of* $\mathrm{Gal}(F/\mathbf{Q})$ *on $C^\bullet(K_F, \mathscr{D}_\kappa)$ and if $\mathscr{D}_\kappa$ admits a slope-$\leq h$ decomposition, the action of* $\mathrm{Gal}(F/\mathbf{Q})$ *stabilizes $C^\bullet(K_F, \mathscr{D}_\kappa)_{\leq h}$.*

Proof. Referring to the definition of $\mathscr{D}_\kappa$ for an arbitrary weight κ, we have $\mathscr{D}_\kappa = \varprojlim \mathcal{D}_\kappa^r$, where $\mathcal{D}_\kappa^r$ is the completion of a module $\mathcal{D}_\kappa$ with respect to a norm $|\cdot|$. The module $\mathcal{D}_\kappa$ itself is the continuous dual of the space $\mathcal{A}_\kappa \subset \mathcal{C}(I, R)$ of continuous functions $f : I \to R$ such that $f(gb) = \kappa(b) f(g)$ for all $g \in I$ and $b \in B_0$. It follows that we have a map $\tau : \mathcal{A}_\kappa \to \mathcal{A}_{\tau(\kappa)}$ (since the action of $\mathrm{Gal}(F/\mathbf{Q})$ preserves both I and B_0). If κ is fixed by τ, we obtain a dual action of $\mathrm{Gal}(F/\mathbf{Q})$ on $\mathcal{D}_\kappa$, and hence $\mathcal{D}_\kappa^r$ and $\mathscr{D}_\kappa$.

Since K_F^p is also stable under the action of $\mathrm{Gal}(F/\mathbf{Q})$ and the actions of K_F^p and $\mathrm{Gal}(F/\mathbf{Q})$ on $\mathscr{D}_\kappa$ commute, by functoriality we obtain an action of $\mathrm{Gal}(F/\mathbf{Q})$ on $C^\bullet(K_F, \mathscr{D}_\kappa)$. Moreover, the action of $\mathrm{Gal}(F/\mathbf{Q})$ fixes the Hecke operator U_p, so [JN16, Proposition 2.2.11] implies that the action of $\mathrm{Gal}(F/\mathbf{Q})$ also preserves $C^\bullet(K_F, \mathscr{D}_\kappa)_{\leq h}$. □

Lemma 3.4.4. *Let $\kappa : T_0 \to R^\times$ be a weight fixed by* $\mathrm{Gal}(F/\mathbf{Q})$. *For any $T \in \mathbb{T}_F^S$, we have $\tau \cdot T = \tau \circ T \circ \tau^{-1}$ as operators on $C^\bullet(K_F, \mathscr{D}_\kappa)$.*

Proof. We may assume $T = [K_F \delta K_F]$ for some $\delta \in \Delta$. Then $\tau \cdot [K_F \delta K_F] = [K_F \tau(\delta) K_F]$, and the corresponding morphism

$$C^\bullet(K_F, \mathscr{D}_\kappa) \to C^\bullet(\tau(\delta) K_F \tau(\delta)^{-1}, \mathscr{D}_\kappa)$$

is induced by the conjugation map $\tau(\delta) K_F \tau(\delta)^{-1} \to K_F$ and the map $\mathscr{D}_\kappa \to \mathscr{D}_\kappa$ given by $d \mapsto \tau(\delta) \cdot d$. But $\tau(\delta) K_F \tau(\delta)^{-1} = \tau\left(\delta \tau^{-1}(K_F) \delta^{-1}\right)$, so we may factor the conjugation map as

$$\tau(\delta) K_F \tau(\delta)^{-1} \xrightarrow{\tau^{-1}} \delta \tau^{-1}(K_F) \delta^{-1} \to \tau^{-1}(K_F) \xrightarrow{\tau} K_F$$

Similarly, $d \mapsto \tau(\delta) \cdot d$ factors as $\tau \circ T \circ \tau^{-1}$, so our morphism of complexes also factors as desired. □

We may restrict $\mathscr{M}_c^*$ to $\mathscr{Z}^{\mathrm{Gal}(F/Q)}$; we denote this restriction by $\mathscr{H}^*$ and by abuse of notation, we again use $\mathscr{T}$ to denote the sheaf generated by the image of $\mathbb{T}_F^S$ in $\mathscr{E}nd_{\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}}(\mathscr{H}^*)$. Then the slice of the eigenvariety $\mathscr{X}^S_{\mathrm{GL}_2/F}$ over $\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}$ is, by definition, $\underline{\mathrm{Spa}}\mathscr{T}$.

Both $\mathbb{T}(\Delta^p, K_F^p)$ and $\mathrm{End}_{\mathscr{O}(V)}(\mathscr{H}_c^*)$ have actions of $\mathrm{Gal}(F/\mathbf{Q})$, and Lemma 3.4.4 implies that they are compatible. Thus, $\mathscr{T}(V)$ and $\mathscr{X}^S_{\mathrm{GL}_2/F}|_{\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}}$ have actions of $\mathrm{Gal}(F/\mathbf{Q})$.

The subspace of $\mathscr{X}_{\mathrm{GL}_2/F}$ fixed by $\mathrm{Gal}(F/\mathbf{Q})$ corresponds to the sheaf $V \mapsto \mathscr{T}(V)_{\mathrm{Gal}(F/\mathbf{Q})}$ of co-invariants of $\mathscr{T}$; by definition, $\mathscr{T}(V)_{\mathrm{Gal}(F/\mathbf{Q})}$ acts on $(\mathscr{H}^*)^{\mathrm{Gal}(F/\mathbf{Q})}$, and the map $\mathscr{T}(V)_{\mathrm{Gal}(F/\mathbf{Q})} \to \mathscr{E}nd_{\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}}\left((\mathscr{H}^*)^{\mathrm{Gal}(F/\mathbf{Q})}\right)$ is injective. Moreover, since $\mathrm{Gal}(F/\mathbf{Q})$ is a finite group with order prime to p, the formation of $(\mathscr{H}^*)^{\mathrm{Gal}(F/\mathbf{Q})}$ commutes with specialization on $\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}$.

The above discussion gives us an eigenvariety datum

$$(\mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}, (\mathscr{H}^*)^{\mathrm{Gal}(F/\mathbf{Q})}, (\mathbb{T}_F^S)_{\mathrm{Gal}(F/\mathbf{Q})}, \psi)$$

and we let $\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q})}_{\mathrm{GL}_2/F}$ denote the corresponding pseudorigid space.

Proposition 3.4.5. *There is a closed immersion $\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q})}_{\mathrm{GL}_2/F} \hookrightarrow \mathscr{X}^S_{\mathrm{GL}_2/F}$, and the image of the morphism $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q},\mathrm{cusp}} \to \mathscr{X}_{\mathrm{GL}_2/F}$ constructed in* [JN19a, §4.3] *is contained in the image of $\mathscr{X}^{\mathrm{Gal}(F/\mathbf{Q})}_{\mathrm{GL}_2/F}$.*

Proof. Both assertions follow from [JN19a, Theorem 3.2.1]. The first follows because $\mathbb{T}(\Delta^p, K^p) \twoheadrightarrow \mathbb{T}(\Delta^p, K^p)_{\mathrm{Gal}(F/\mathbf{Q})}$ is a surjection. The second follows because classical points are very Zariski-dense in $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q}}$, and the fact that the image of a classical system of Hecke eigenvalues under the classical cyclic base change map is fixed by $\mathrm{Gal}(F/\mathbf{Q})$; since $(\mathscr{H}^*(z))^{\mathrm{Gal}(F/\mathbf{Q})} = (\mathscr{H}^*)^{\mathrm{Gal}(F/\mathbf{Q})}(z)$ for all $z \in \mathscr{Z}^{\mathrm{Gal}(F/\mathbf{Q})}$, we may again apply [JN19a, Theorem 3.2.1]. $\square$

We let

$$\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}_{\mathrm{GL}_2/F} := \mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q})}_{\mathrm{GL}_2/F} \cap \mathscr{X}^S_{\mathrm{GL}_2/F,\mathrm{mid}}$$

and we let $\overline{\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}_{\mathrm{GL}_2/F}}$ denote its Zariski closure in $\mathscr{X}^S_{\mathrm{GL}_2/F}$.

Lemma 3.4.6. *Classical points are very Zariski dense in $\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}_{\mathrm{GL}_2/F}$.*

Proof. If (U, h) is a slope datum and $W \subset \mathscr{X}^S_{\mathrm{GL}_2/F}$ is a connected affinoid subspace of the pre-image of U, then $\mathscr{T}(W) = e_W\mathscr{T}(U)$ and $\mathscr{M}_c^*(W) \cong e_W H_c^*(K, \mathscr{D}_U)_{\leq h}$, where e_W is the idempotent projector to W. If $W \subset \mathscr{X}^S_{\mathrm{GL}_2/F,\mathrm{mid}}$, then $\mathscr{M}_c^* \cong e_W H_c^d(K, \mathscr{D}_U)_{\leq h}$ and $H_c^d(K, \mathscr{D}_U)_{\leq h}$ is a projective $\mathscr{O}_{\mathscr{W}}(U)$-module. It follows that the restriction of $\mathscr{M}_c^*$ to $\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}_{\mathrm{GL}_2/F}$ is a vector bundle, and since $|\mathrm{Gal}(F/\mathbf{Q})|$ is prime to p, its $\mathrm{Gal}(F/\mathbf{Q})$-invariants remain projective.

Now we may apply [Che04, Lemme 6.2.10] to conclude that $\mathscr{T}(W)$ is equidimensional of dimension $\dim \mathscr{O}_{\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}}(U)$, and every irreducible component of $\operatorname{Spec} \mathscr{T}(W)$ surjects onto an irreducible component of $\operatorname{Spec} \mathscr{O}_{\mathscr{W}_F^{\mathrm{Gal}(F/\mathbf{Q})}}(U)$. If $x \in W$ has a classical weight that is sufficiently large (where "sufficiently large" depends on h), then x corresponds to a classical Hilbert modular form. But sufficiently large classical weights are Zariski dense in U, so [Che04, Lemme 6.2.8] implies that classical points are dense in W. □

Remark 3.4.7. The proofs of Proposition 3.4.5 and Lemma 3.4.6 are the only times we use our assumption that $|\mathrm{Gal}(F/\mathbf{Q})|$ is prime to p. If we restricted to the rigid analytic locus (where p is invertible, so $(\mathscr{H}^*)^{\mathrm{Gal}(F/\mathbf{Q})}$ is unconditionally projective, with $(\mathscr{H}^*(z))^{\mathrm{Gal}(F/\mathbf{Q})} = (\mathscr{H}^*)^{\mathrm{Gal}(F/\mathbf{Q})}(z))$, this hypothesis would be unnecessary.

Corollary 3.4.8. *The image of the cyclic base change morphism in $\mathscr{X}^S_{\mathrm{GL}_2/F,\mathrm{mid}}$ is exactly $\overline{\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}_{\mathrm{GL}_2/F}}$.*

Proof. Since the morphism $\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}_{\mathrm{GL}_2/F} \to \mathscr{X}_{\mathrm{GL}_2/F}$ is finite, it has closed image. Moreover, cyclic base change carries any classical point of $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q},\mathrm{cusp}}$ to a point of $\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}_{\mathrm{GL}_2/F}$. On the other hand, every classical point of $\mathscr{X}^{S,\mathrm{Gal}(F/\mathbf{Q}),\circ}_{\mathrm{GL}_2/F}$ is in the image of cyclic base change, by the classical theorem, so Lemma 3.4.6 implies the desired result. □

3.5. **Galois representations.** In [JN16, §5.4], the authors construct families of Galois determinants (in the sense of [Che14]) over the eigenvarieties $\mathscr{X}_{\mathbf{G}}$ when $\mathbf{G} = \operatorname{Res}_{F/\mathbf{Q}} \mathrm{GL}_n$ and F is totally real or CM, and prove that they satisfy local-global compatibility at places away from p and the level. Then the Jacquet–Langlands correspondence lets us deduce the following:

Theorem 3.5.1. *Let D be a quaternion algebra over a totally real field F, such that F is totally split at p and D is split at all places above p. Let $K = K^pI \subset (\mathbf{A}_{F,f} \otimes D)^\times$ be the level, and let S be the set of finite places v of F for which D is ramified or $K_v \neq \mathrm{GL}_2(\mathscr{O}_{F_v})$. Then there is a continuous 2-dimensional Galois determinant $D : \mathrm{Gal}_{F,S} \to \mathscr{O}(\mathscr{X}_{\underline{D}^\times})^+$ such that*

$$D(1 - X \cdot \mathrm{Frob}_v) = P_v(X)$$

for all $v \notin S$, where $P_v(X)$ is the standard Hecke polynomial.

Moreover, if $v \mid p$ then for every maximal point $x \in \mathscr{X}_{\underline{D}^\times}$ of weight $\kappa_x = (\kappa_{x,1}, \kappa_{x,2})$, we let $\psi : \mathscr{O}(\mathscr{X}_{\underline{D}^\times})^+ \to k(x)^+$ denote the corresponding specialization map. Then the Galois representation corresponding to $D_x|_{\mathrm{Gal}_{F,v}}$ is trianguline with parameters $\delta_1, \delta_2 : F_v^\times \rightrightarrows k(v)^\times$, where

$$\delta_1|_{\mathscr{O}_{F_v}^\times} = \kappa_{x,2}^{-1}|_{\mathscr{O}_v^\times} \text{ and } \delta_1(\varpi_v) = \psi(U_v)$$

and

$$\delta_2|_{\mathscr{O}_{F_v}^\times} = (\kappa_{x,1}|_{\mathscr{O}_v^\times}\chi_{\mathrm{cyc}})^{-1} \text{ and } \delta_2(\varpi_v) = \psi(I_v \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right) I_v)$$

Proof. It only remains to check local-global compatibility at places above p. But this is true for classical points by work of Saito, Blasius–Rogawski, and Skinner, and it is true for twist classical points by the definition of twisting. Then the result follows from [Bel21, Theorem 6.3.1]. □

Remark 3.5.2. For each point $x \in \mathscr{X}_{\underline{D}^\times}$, there is a residual Galois determinant $\overline{D}_x$ valued in a finite field. These residual Galois determinants are constant on each connected component of $\mathscr{X}_{\underline{D}^\times}$, as a consequence of [Che14, Lemma 3.10].

3.6. **Quaternionic sub-eigenvarieties.** In order to study suitable spaces of overconvergent quaternionic modular forms, we will need to define and study eigenvarieties parametrizing quaternionic modular forms with certain auxiliary data fixed. We let F be a totally real number field totally split at p, and we let D be a totally definite quaternion algebra over F, split at all places above p. We fix a level $K \subset (\mathbf{A}_{F,f} \otimes_F D)^\times$ and monoid $K \subset \Delta \subset (\mathbf{A}_{F,f} \otimes_F D)^\times$, and we set $\mathbb{T}$ to be either $\mathbb{T}(\Delta^p, K^p)$ or $\mathbb{T}(\Delta, K)$.

In order to construct an eigenvariety for $\underline{D}$, we fixed a Borel–Serre complex $C_c^\bullet(K,-)$ and we considered the cohomology $C_c^\bullet(K, \mathscr{D}_\kappa)$. However, because we assumed D is totally definite, the associated Shimura manifold is a finite set of points, and so the cohomology of $C_c^\bullet(K,-) = C^\bullet(K,-)$ vanishes outside of degree 0.

Thus, we can give an extremely concrete description of the automorphic forms of interest to us and of the Hecke operators acting on them. Suppose that M is a left $R[\Delta]$-module, for some pseudoaffinoid algebra R. Then if $f : D^\times\backslash(\mathbf{A}_{F,f} \otimes_F D)^\times \to M$ is a function and $\gamma \in \Delta$, we define ${}_\gamma|f$ via ${}_\gamma|f(g) = \gamma \cdot f(g\gamma)$. Then

$$H^0(K,M) = \left\{f : D^\times\backslash(\mathbf{A}_{F,f} \otimes_F D)^\times \to M \mid {}_\gamma|f = f \text{ for all } \gamma \in K\right\}$$

We can describe the Hecke operator $[KgK] : H^0(K,M) \to H^0(K,M)$ explicitly for any $g \in \Delta$; we decompose the double coset $KgK = \coprod_i g_iK$ as a finite disjoint union of cosets, and we have

$$[KgK]f := \sum_i {}_{g_i}|f$$

The first piece of auxiliary data we want to fix is the central character. If $\xi : \mathbf{A}_{F,f}^\times/F^\times \to R_0^\times$ is a continuous character such that $\xi|_{K_v\cap\mathscr{O}_{F_v}^\times}$ agrees with the action of $K_v \cap \mathscr{O}_{F_v}^\times$ on M for all finite places v of F, we may extend the action of K on M to an action of $K \cdot \mathbf{A}_{F,f}^\times$, by letting $\mathbf{A}_{F,f}^\times$ act by ξ. Then we define

$$H^0(K,M)[\xi] := \{f \in H^0(K,M) \mid {}_z|f = f \text{ for all } z \in \mathbf{A}_{F,f}^\times\}$$

If we write $D^\times\backslash(\mathbf{A}_{F,f}\otimes_F D)^\times/K = \coprod_{i\in I} D^\times g_i K \mathbf{A}_{F,f}^\times$ for some finite set of elements $g_i \in (\mathbf{A}_{F,f}\otimes_F D)^\times$, the natural map

$$\begin{aligned} H^0(K,M)[\xi] &\to \oplus_{i\in I} M^{(K\mathbf{A}_{F,f}^\times \cap g_i^{-1}D^\times g_i)/F^\times} \\ f &\mapsto (f(g_i)) \end{aligned}$$

is an isomorphism.

The calculations of [Tay06, Lemma 1.1] show that $(K\mathbf{A}_{F,f}^\times \cap g_i^{-1}D^\times g_i)/F^\times$ is a finite group with order prime to p for all i (since we assumed $p \neq 2$). Thus, if M is a potentially orthonormalizable Banach R-module, then so is $H^0(K,M)[\xi]$, and we will be able to apply the formalism of slope decompositions to quaternionic modular forms with fixed central character. More precisely, we may define the Fredholm characteristic power series F_ξ of a compact operator U on $H^0(K,M)[\xi]$; if F_ξ admits a factorization $F_\xi = Q_\xi S_\xi$ with Q_ξ a multiplicative polynomial and S_ξ a Fredholm series, then $K_\xi := \ker\left(Q_\xi^*(U)\right)$ is a projective R-module, by [JN16, Theorem 2.2.2].

The coefficient modules of interest to us are the modules of distributions $\mathscr{D}_\kappa$ constructed in [JN16], and we fix a character $\xi : \mathbf{A}_{F,f}^\times/F^\times \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$ as above. The operator U_p commutes with the action of $\mathbf{A}_{F,f}^\times/F^\times$ on $\mathscr{D}_\kappa$ given by ξ, so U_p acts compactly on $C^*(K,\mathscr{D}_\kappa)[\xi]$. We may construct a corresponding spectral variety $\mathscr{Z}_\xi$ and eigenvariety datum $(\mathscr{Z}_\xi, \mathscr{M}_\xi, \mathbb{T}, \psi)$, where $\mathscr{M}_\xi$ is the coherent sheaf on $\mathscr{Z}_\xi$ coming from factorizations of the characteristic power series of U_p; we write $\mathscr{X}_{\underline{D}^\times,\xi}$ for the corresponding eigenvariety.

By construction, $H^0(K,\mathscr{D}_\kappa)[\xi]_{\leq h}$ is a projective R-module whenever (U,h) is a slope datum. Then [Che04, Lemme 6.2.10] implies that if $\mathscr{M}_\xi$ is non-zero, $\mathscr{X}_{\underline{D}^\times,\xi}$ is equidimensional of the same dimension as $\mathscr{W}_F$.

Moreover, for each maximal point $x \in \mathscr{X}_{\underline{D}^\times,\xi}$, the corresponding Hecke eigensystem appears in $\mathscr{X}_{\underline{D}^\times}$ (with unrestricted central character), by construction. Then the interpolation theorem [JN19a, Theorem 3.2.1] implies that there is a closed immersion $\mathscr{X}_{\underline{D}^\times,\xi}^{\mathrm{red}} \hookrightarrow \mathscr{X}_{\underline{D}^\times}$, and dimension considerations imply that its image is a union of irreducible components of $\mathscr{X}_{\underline{D}^\times}$.

This implies in particular that as (U,h) runs over slope data for $C^*(K,\mathscr{D}_\kappa)[\xi]$, the sets $W_{U,h}^{\prime,\mathrm{ss}} \subset U$ of semi-simple weights constructed in Proposition 3.3.5 are Zariski dense. Then we may repeat the argument of that proposition to conclude that $\mathscr{X}_{\underline{D}^\times,\xi}$ is itself reduced.

We have shown the following:

Proposition 3.6.1. *Given a character $\xi : \mathbf{A}_{F,f}^\times/F^\times \to \mathscr{O}(\mathscr{W}_F)^\times$ as above, there is an eigenvariety $\mathscr{X}_{\underline{D}^\times,\xi}$ of quaternionic modular forms with central character ξ. It is reduced and equidimensional, and it is naturally identified as a (possibly empty) union of irreducible components of $\mathscr{X}_{\underline{D}^\times}$.*

We also wish to introduce eigenvarieties localized at maximal ideals of Hecke algebras. Let $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal. By Theorem 3.5.1 and Remark 3.5.2, the residual Hecke eigenvalues are locally constant on $\mathscr{X}_{\underline{D}^\times}$. It follows that the restrictions $\mathscr{M}_\mathfrak{m}$ and $\mathscr{M}_{\xi,\mathfrak{m}}$ are supported on unions of connected components of $\mathscr{Z}$, which we write $\mathscr{Z}_\mathfrak{m}$ and $\mathscr{Z}_{\xi,\mathfrak{m}}$, respectively. In particular, if (U, h) is a slope datum, then $H^0(K, \mathscr{D}_U)_{\le h,\mathfrak{m}}$ and $H^0(K, \mathscr{D}_U)_{\le h,\xi,\mathfrak{m}}$ are again finite projective $\mathscr{O}(U)$-modules. Then an identical argument shows the following:

Proposition 3.6.2. *Given a character $\xi : \mathbf{A}_{F,f}^\times/F^\times \to \mathscr{O}(\mathscr{W}_F)^\times$ as above and a maximal ideal $\mathfrak{m} \subset \mathbb{T}$ as above, for any choice of Hecke algebra $\mathbb{T}'$ (possibly different from $\mathbb{T}$) there are eigenvarieties $\mathscr{X}^{\mathbb{T}'}_{\underline{D}^\times,\mathfrak{m}}$ and $\mathscr{X}^{\mathbb{T}'}_{\underline{D}^\times,\xi,\mathfrak{m}}$ of quaternionic modular forms localized at $\mathfrak{m}$. They are reduced and equidimensional, and they are naturally identified as (possibly empty) unions of connected components of $\mathscr{X}^{\mathbb{T}'}_{\underline{D}^\times}$.*

Remark 3.6.3. We write $h = m/n$ and consider the closed ball $\mathbb{B}_{U,h} := \{|T^n| \le |u^{-m}|\} \subset \mathbb{A}^1_U$ for some open affinoid $U \subset \mathscr{W}_F$. Setting $Z_{U,h} := \mathscr{Z}_\mathfrak{m} \cap \mathbb{B}_{U,h}$ (resp. $Z_{U,h} := \mathscr{Z}_{\xi,\mathfrak{m}} \cap \mathbb{B}_{U,h}$), we abuse terminology slightly and say that (U, h) is a slope datum for $\mathscr{X}_{\underline{D}^\times,\mathfrak{m}}$ (resp. $\mathscr{X}_{\underline{D}^\times,\xi,\mathfrak{m}}$) if $Z_{U,h} \to U$ is finite of constant degree.

4. Overconvergent quaternionic modular forms

4.1. **Definitions.** We will use overconvergent cohomology to define and study spaces of overconvergent quaternionic modular forms. Maintaining our notation from § 3.1, and in particular § 3.6, we fix a level $K \subset (\mathbf{A}_{F,f} \otimes_F D)^\times$ and monoid $K \subset \Delta \subset (\mathbf{A}_{F,f} \otimes_F D)^\times$, and we set $\mathbb{T}$ to be either $\mathbb{T}(\Delta^p, K^p)$ or $\mathbb{T}(\Delta, K)$.

The coefficients for our families of overconvergent modular forms will be a pseudoaffinoid algebra R over $\mathbf{Z}_p$; we set $U := \operatorname{Spa} R$. We also fix a pseudouniformizer $u \in R$. If $\kappa : T_0/\overline{Z(K)} \to R^\times$ is a weight, we choose a norm $|\cdot|$ on R so that $|\cdot|$ is adapted to κ and multiplicative with respect to u, and $\log_p|\cdot|$ is discrete (which we may do, by Lemma 4.1.1 below). Then the unit ball $R_0 \subset R$ is a ring of definition containing u.

Fix some $r \ge r_\kappa$. We let $\mathcal{D}_\kappa^{r,\circ} \subset \mathcal{D}_\kappa^r$ denote the unit ball, and we also consider larger modules of distributions $\mathcal{D}_\kappa^{<r} \supset \mathcal{D}_\kappa^r$, with unit ball $\mathcal{D}_\kappa^{<r,\circ} \subset \mathcal{D}_\kappa^{<r}$. Following §3.6, we also fix a character $\xi : \mathbf{A}_{F,f}^\times/F^\times \to R^\times$ such that $\xi|_{K_v \cap \mathscr{O}_{F_v}^\times}$ agrees with the action of $K_v \cap \mathscr{O}_{F_v}^\times$ on $\mathcal{D}_\kappa^r$, that is, such that $\xi|_{K_v \cap \mathscr{O}_{F_v}^\times}$ is trivial for $v \nmid p$ and $\xi|_{I_v \cap \mathscr{O}_{F_v}^\times}$ is equal to the action of $I_v \cap \mathscr{O}_{F_v}^\times$ on $\mathcal{D}_\kappa^r$ for $v \mid p$.

The construction of the required norm on R is a variant of [JN16, Lemma 3.3.1], and we refer to that paper for the terminology:

Lemma 4.1.1. *If R is a pseudoaffinoid algebra over $\mathbf{Z}_p$ and $\kappa : T_0/\overline{Z(K)}$ is a weight, there is a norm $|\cdot|$ on R such that $|\cdot|$ is adapted to κ and multiplicative with respect to u, the unit ball R_0 is noetherian, and $\log_p|\cdot|$ is discrete.*

Proof. Choose a noetherian ring of definition $R_0 \subset R$ formally of finite type over $\mathbf{Z}_p$. As in the proof of [JN16, Lemma 3.3.1], $\kappa(T_0) \subset R^\circ$ and $\kappa(T_\epsilon) \subset 1+R^{\circ\circ}$; since both groups are topologically finitely generated, we may replace R_0 with a finite integral extension and assume that $\kappa(T_0) \subset R_0$, and we may find some integer $m \geq 1$ so that $\kappa(T_\epsilon)^m \subset 1 + uR_0$.

Let $R' := R[u^{1/m}]$, let $R_0' := R_0[u^{1/m}]$, and let $u' := u^{1/m}$. Then R' is a finite R-module, so it has a canonical topology, and the subspace topology it induces on R agrees with the original topology on R. Now for any $a \in \mathbf{R}_{>1}$ we may define a norm $|\cdot|'$ on R' via

$$|r'|' = \inf\{a^s \mid {u'}^s r' \in R_0'\}$$

The restriction of $|\cdot|'$ to R has the desired properties. $\square$

Recall that we have Fredholm power series

$$F_\kappa := \det\left(1 - TU_p \mid H^0(K, \mathcal{D}_\kappa^r)\right)$$

and

$$F_{\kappa,\xi} := \det\left(1 - TU_p \mid H^0(K, \mathcal{D}_\kappa^r)[\xi]\right)$$

and they are independent of $r \geq r_\kappa$, by [JN16, Proposition 4.1.2]. If F_κ (resp. $F_{\kappa,\xi}$) has a slope $\leq h$-factorization, then the formalism of slope decompositions implies that we have a decomposition

$$H^0(K, \mathcal{D}_\kappa^r) = H^0(K, \mathcal{D}_\kappa^r)_{\leq h} \oplus H^0(K, \mathcal{D}_\kappa^r)_{>h}$$

resp.

$$H^0(K, \mathcal{D}_\kappa^r)[\xi] = H^0(K, \mathcal{D}_\kappa^r)[\xi]_{\leq h} \oplus H^0(K, \mathcal{D}_\kappa^r)[\xi]_{>h}$$

for all $r \geq r_\kappa$, and the decomposition is independent of r.

Moreover, if $r' \in [r_\kappa, r)$, the inclusions

$$\mathcal{D}_\kappa^r \subset \mathcal{D}_\kappa^{<r} \subset \mathcal{D}_\kappa^{r'}$$

induce an isomorphism $H^0(K, \mathcal{D}_\kappa^r)_{\leq h} \xrightarrow{\sim} H^0(K, \mathcal{D}_\kappa^{r'})_{\leq h}$. We may therefore define

$$H^0(K, \mathcal{D}_\kappa^{<r})_{\leq h} := \operatorname{im}\left(H^0(K, \mathcal{D}_\kappa^r)_{\leq h} \to H^0(K, \mathcal{D}_\kappa^{<r})\right)$$

and

$$H^0(K, \mathcal{D}_\kappa^{<r})[\xi]_{\leq h} := \operatorname{im}\left(H^0(K, \mathcal{D}_\kappa^r)[\xi]_{\leq h} \to H^0(K, \mathcal{D}_\kappa^{<r})[\xi]\right)$$

As before,

We make the additional definitions

$$H^0(K, \mathcal{D}_\kappa^{<r,\circ})_{\leq h} := \operatorname{im}\left(H^0(K, \mathcal{D}_\kappa^{<r,\circ}) \to H^0(K, \mathcal{D}_\kappa^{<r}) \to H^0(K, \mathcal{D}_\kappa^{<r})_{\leq h}\right)$$

and

$$H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h} := \operatorname{im}\left(H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi] \to H^0(K, \mathcal{D}_\kappa^{<r})[\xi] \to H^0(K, \mathcal{D}_\kappa^{<r})[\xi]_{\leq h}\right)$$

We are now in a position to define spaces of overconvergent quaternionic modular forms, together with an integral structure and Hecke algebras:

Definition 4.1.2. Suppose that $H^0(K, \mathscr{D}_\kappa)$ admits a slope-$\leq h$ decomposition. We define the modular forms of weight κ and slope-$\leq h$ to be the module $S_\kappa(K)_{\leq h} := H^0(K, \mathscr{D}_\kappa)_{\leq h}$; it is a module over the Hecke algebra $\mathbb{T}_{\kappa,\leq h} := \operatorname{im}\left(\mathbb{T} \otimes_{\mathbf{Z}_p} R \to \operatorname{End}_R(S_\kappa(K)_{\leq h})\right)$.

We define integral modular forms of weight κ and slope-$\leq h$ to be the R_0-submodule $S_\kappa^{<r,\circ}(K)_{\leq h} := H^0(K, \mathcal{D}_\kappa^{<r,\circ})_{\leq h}$, and it is a module over the integral Hecke algebra $\mathbb{T}^{<r,\circ}_{\kappa,\leq h} := \operatorname{im}\left(\mathbb{T} \otimes_{\mathbf{Z}_p} R_0 \to \operatorname{End}_{R_0}(S_\kappa^\circ(K)_{\leq h})\right)$.

If $\xi : \mathbf{A}_{F,f}^\times / F^\times \to R_0^\times$ is a continuous character as above and $H^0(K, \mathscr{D}_\kappa^r)[\xi]$ admits a slope-$\leq h$ decomposition, we define the modular forms with central character ξ to be $S_{\kappa,\xi}(K)_{\leq h} := H^0(K, \mathscr{D}_\kappa)[\xi]_{\leq h}$ and similarly for integral modular forms with central character ξ.

We now fix a choice of Hecke algebra. Let S denote the set of places of F such that $v \mid p$, D is ramified at v, or $K_v \neq \mathcal{O}_{D,v}^\times$. For $v \notin S$, we define

$$S_v := \left[K \begin{pmatrix} \varpi_v & \\ & \varpi_v \end{pmatrix} K\right], \quad T_v := \left[K \begin{pmatrix} 1 & \\ & \varpi_v \end{pmatrix} K\right] \in K \backslash (\mathbf{A}_{F,f} \otimes D)^\times / K$$

for some fixed uniformizer ϖ_v of $\mathscr{O}_{F_v}$.

We define the Hecke algebra $\mathbb{T}$ to be the free commutative $\mathbf{Z}_p$-algebra generated by $\{U_v\}_{v \mid p}$ and $\{S_w, T_w\}_{w \notin S}$. Since Δ_p acts on the modules of distributions $\mathcal{D}_\kappa^{<r,\circ}$ and Hecke operators away from p preserve the slope decomposition, we may view $S_\kappa^{<r,\circ}(K)_{\leq h}$ as a $\mathbb{T}$-module.

We also describe the so-called diamond operators, after modifying the tame level K^p. Suppose we have a finite set Q of places of F such that for each $v \in Q$, $v \nmid p$, $\operatorname{Nm} v \equiv 1 \pmod p$, D is split at v, and $K_v = \mathrm{GL}_2(\mathscr{O}_{F_v})$. For each $v \in Q$, we again let $K_0(v) \subset \mathrm{H}(F_v)$ denote the subgroup $\{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \mod v\}$, and we consider the homomorphism

$$K_0(v) \to k(v)^\times \to \Delta_v$$

given by composing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad^{-1}$$

with the projection to the p-power subgroup $\Delta_v \subset k(v)^\times$. Let $K^-(v)$ denote the group

$$K^-(v) := \left\{\begin{pmatrix} a & * \\ 0 & d \end{pmatrix} \mid ad^{-1} \mapsto 1 \text{ in } \Delta_v\right\}$$

for each $v \in Q$, and let

$$K_0(Q) := \prod_{v \in Q} K_0(v) \cdot \prod_{v \notin Q} K_v$$

and

$$K^-(Q) := \prod_{v \in Q} K^-(v) \cdot \prod_{v \notin Q} K_v$$

Then $K_0(v)/K^-(v) \cong \Delta_v$, and every $h \in \Delta_Q := \prod_{v\in Q}\Delta_v$ gives rise to a Hecke operator

$$\langle h\rangle := \left[K^-(Q)\widetilde{h}K^-(Q)\right]$$

on $S_\kappa^{<r,\circ}(K^-(Q))$, where $\widetilde{h}$ is a lift of h to $K_0(Q)$; $\langle h\rangle$ is independent of the choice of $\widetilde{h}$.

We let $\mathbb{T}_Q^-$ be the free commutative $\mathbf{Z}_p$-algebra generated by $\{U_v\}_{v|p}$, $\{S_v, T_v\}_{v\notin S}$, and $\{U_{\varpi_v}\}_Q$, where $U_{\varpi_v} := [K^-(v)\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi_v\end{smallmatrix}\right)K^-(v)]$; it acts naturally on $S_\kappa^{<r,\circ}(K^-(Q))_{\le h}$, and we let $\mathbb{T}^{<r,\circ}_{K^-(Q),\le h}$ denote the R_0-algebra its image generates in $\mathrm{End}_{R_0}(S_\kappa^{<r,\circ}(K^-(Q))_{\le h})$. Similarly, we let $\mathbb{T}_{0,Q}$ be the free commutative $\mathbf{Z}_p$-algebra generated by $\{U_v\}_{v|p}$, $\{S_v,T_v\}_{v\notin S}$, and $\{U_{\varpi_v}\}_Q$, where $U_{\varpi_v} := [K_0(v)\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi_v\end{smallmatrix}\right)K_0(v)]$.

4.2. **Varying the level.** We record some results on the existence of slope decompositions as we vary the tame level. Fix a set of places Q as above, and fix a maximal ideal $\mathfrak{m} \subset \mathbb{T}$ which corresponds to the residual Hecke eigenvalues at some maximal point of $\mathscr{X}_{\underline{D}^\times}$. There is a corresponding Galois representation $\overline{\rho}_{\mathfrak{m}} : \mathrm{Gal}_F \to \mathrm{GL}_2(\mathbf{F})$ for some finite field $\mathbf{F}$; it is unramified at all places of Q and the characteristic polynomial of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_v)$ is $X^2 - T_vX + \mathrm{Nm}(v)S_v$ for all $v \in Q$. After replacing $\mathbf{F}$ with a quadratic extension if necessary, we may assume that each such characteristic polynomial has roots $\{\alpha_v, \beta_v\}$ in $\mathbf{F}$; we assume that $\alpha_v\beta_v^{-1} \notin \{1, \mathrm{Nm}(v)^{\pm}\}$.

Let $E/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathscr{O}_E$ and residue field containing $\mathbf{F}$, and replace the Hecke algebras $\mathbb{T}$ and $\mathbb{T}_{Q,0}$ with $\mathscr{O}_E \otimes_{\mathbf{Z}_p} \mathbb{T}$ and $\mathscr{O}_E \otimes_{\mathscr{O}_E} \mathbb{T}_{Q,0}$, respectively. Similarly, replace the coefficient module $\mathscr{D}_\kappa$ with its base-change to $\mathscr{O}_E$, so that the Hecke algebras continue to act (the upshot is that we also base-change the resulting eigenvarieties from $\mathbf{Z}_p$ to $\mathscr{O}_E$, but we suppress this from the notation). Fix a root $\alpha_v \in \mathbb{F}$ of each characteristic polynomial, and fix a lift $A_v \in \mathscr{O}_E$ of each α_v. Then we define $\mathfrak{m}_{Q,0} \subset \mathbb{T}_{Q,0}$ to be the maximal ideal generated by $\mathfrak{m} \cap \mathbb{T}_{Q,0}$ and $U_v - A_v$ for all $v \in Q$.

Lemma 4.2.1. *Fix a central character $\xi : \mathbf{A}_{F,f}^\times/F^\times \to \mathscr{O}(\mathscr{W}_F)^\times$. Then there is an isomorphism $\mathscr{X}^{K,\mathbb{T}_{Q,0}}_{\underline{D}^\times,\xi,\mathfrak{m}} \xrightarrow{\sim} \mathscr{X}^{K_Q(0),\mathbb{T}_{Q,0}}_{\underline{D}^\times,\xi,\mathfrak{m}_{Q,0}}$, compatible with the respective morphisms to $\mathscr{W}_F$.*

Proof. Let $S_\kappa(K)_{\le h,\mathfrak{m}} := \mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{T}} S_\kappa(K)_{\le h}$, and similarly for $S_\kappa(K_Q(0))_{\le h,\mathfrak{m}_{Q,0}}$. By [Kis09a, Lemma 2.1.7], for any slope h and any sufficiently large classical weight κ, we have an isomorphism of $\mathbb{T}_{Q,0}$-modules

$$S_\kappa(K)_{\le h,\mathfrak{m}} \xrightarrow{\sim} S_\kappa(K_Q(0))_{\le h,\mathfrak{m}_{Q,0}}$$

By construction, classical points are dense in $\mathscr{X}^{K,\mathbb{T}_{Q,0}}_{\underline{D}^\times,\xi,\mathfrak{m}}$ and $\mathscr{X}^{K_Q(0),\mathbb{T}_{Q,0}}_{\underline{D}^\times,\xi,\mathfrak{m}_{Q,0}}$, so we may use [JN19a, Theorem 3.2.1] to construct morphisms of eigenvarieties

$$\mathscr{X}^{K_Q(0),\mathbb{T}_{Q,0}}_{\underline{D}^\times,\xi,\mathfrak{m}_{Q,0}} \to \mathscr{X}^{K,\mathbb{T}_{Q,0}}_{\underline{D}^\times,\xi,\mathfrak{m}}$$

and

$$\mathscr{X}^{K,\mathbb{T}_{Q,0}}_{\underline{D}^\times,\xi,\mathfrak{m}} \to \mathscr{X}^{K_Q(0),\mathbb{T}_{Q,0}}_{\underline{D}^\times,\xi,\mathfrak{m}_{Q,0}}$$

These morphisms are mutually inverse, so they are isomorphisms. □

Corollary 4.2.2. *Fix a central character $\xi : \mathbf{A}^\times_{F,f}/F^\times \to \mathscr{O}(\mathscr{W}_F)^\times$. Let $U = \operatorname{Spa} R \subset \mathscr{W}_F$ be an affinoid open, corresponding to a weight κ, and fix $h \in \mathbb{Q}_{>0}$. Then (U,h) is a slope datum for $\mathscr{X}^{K}_{\underline{D}^\times,\xi,\mathfrak{m}}$ if and only if it is a slope datum for $\mathscr{X}^{K_0(Q)}_{\underline{D}^\times,\xi,\mathfrak{m}_{Q,0}}$.*

Lemma 4.2.3. *Fix a central character $\xi : \mathbf{A}^\times_{F,f}/F^\times \to \mathscr{O}(\mathscr{W}_F)^\times$. Let $U = \operatorname{Spa} R \subset \mathscr{W}_F$ be an affinoid open, corresponding to a weight κ, and fix $h \in \mathbb{Q}_{>0}$. Then $H^0(K_0(Q),\mathscr{D}_\kappa)[\xi]$ admits a slope-$\leq h$ decomposition if and only if $H^0(K^-(Q),\mathscr{D}_\kappa)[\xi]$ admits a slope decomposition.*

Proof. For each $h \in \Delta_Q$, we may choose a coset representative $\widetilde{h}$ of $K_0(Q)/K^-(Q)$ of the form $\widetilde{h} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & d_h \end{smallmatrix}\right) \in \prod_{v\in Q} \mathscr{O}_{F_v} \otimes \mathscr{O}_D$, where $d_h \in \mathscr{O}_F$ is a lift of $h \in \Delta_Q$ chosen to be relatively prime to $v \mid p$. Then if $D^\times\backslash(\mathbf{A}_{F,f}\otimes_F D)^\times/K_0(Q) = \coprod_{i\in I} D^\times g_i K$, we have $D^\times\backslash(\mathbf{A}_{F,f}\otimes_F D)^\times/K^-(Q) = \coprod_{i\in I}\coprod_{h\in\Delta_Q} D^\times g_i\widetilde{h}K$ and we have an isomorphism

$$H^0(K^-(Q),\mathscr{D}_\kappa)[\xi] \xrightarrow{\sim} \oplus_{h\in\Delta_Q}\oplus_{i\in I}\mathscr{D}_\kappa^{(K^-(Q)\mathbf{A}_{F,f}\cap(g_i\widetilde{h})^{-1}D^\times(g_i\widetilde{h}))/F^\times}$$
$$f \mapsto \left(f(g_i\widetilde{h})\right) = \left(\left({}_{\widetilde{h}}|f\right)(g_i)\right)$$

Similarly, we have an isomorphism

$$H^0(K_0(Q),\mathscr{D}_\kappa)[\xi] \xrightarrow{\sim} \oplus_{i\in I}\mathscr{D}_\kappa^{(K^-(Q)\mathbf{A}_{F,f}\cap(g_i\widetilde{h})^{-1}D^\times(g_i\widetilde{h}))/F^\times}$$
$$f \mapsto (f(g_i))$$

Grouping terms, we obtain a decomposition

$$H^0(K^-(Q),\mathscr{D}_\kappa)[\xi] \xrightarrow{\sim} \oplus_{h\in\Delta_Q}{}_{\widetilde{h}}|H^0(K_0(Q),\mathscr{D}_\kappa)[\xi]$$

We may also write

$$U_p = \sum_a {}_{U_{p,a}}|$$

where $U_{p,a} := \left(\begin{smallmatrix} 1 & 0 \\ p\widetilde{a} & p \end{smallmatrix}\right)$ and a runs over residue classes modulo p. A computation shows that $\widetilde{h}U_{p,a} = U_{p,ad_h}\widetilde{h}$; we chose the lifts d_h to be prime to p, so $\{ad_h\} = \{a\}$, and so U_p commutes with the operator ${}_{\widetilde{h}}|$ for all $h \in \Delta_Q$.

Thus, the U_p operator on $H^0(K^-(Q),\mathscr{D}_\kappa)[\xi]$ preserves each direct summand ${}_{\widetilde{h}}|H^0(K_0(Q),\mathscr{D}_\kappa)[\xi]$. It follows that the characteristic power series of U_p

acting on $H^0(K^-(Q), \mathscr{D}_\kappa)[\xi]$ admits a slope-$\leq h$ factorization if and only if the characteristic power series of U_p acting on $H^0(K_0(Q), \mathscr{D}_\kappa)[\xi]$ does. $\square$

4.3. **Integral overconvergent quaternionic modular forms.** We need to make a closer study of the structure of the integral modules of distributions and their finite-slope subspaces.

Lemma 4.3.1. *If $\kappa : T_0/\overline{Z(K)} \to R^\times$ is a weight and F_κ has a slope $\leq h$-factorization, then $S_\kappa(K)_{\leq h}$ and $S_{\kappa,\xi}(K)_{\leq h}$ are finite projective R-module, and they are compatible with arbitrary base change on R.*

Proof. The base change spectral sequence of [JN16, Theorem 4.2.1] and the vanishing of overconvergent cohomology in degrees greater than 0 imply that the formation of $H^0(K, \mathscr{D}_\kappa)_{\leq h}$ and $H^0(K, \mathscr{D}_\kappa)[\xi]_{\leq h}$ commute with arbitrary base change on R. But this implies that they are flat. Since $S_\kappa(K)_{\leq h}$ (and hence $S_{\kappa,\xi}(K)_{\leq h}$) is a finite R-module (by [JN16, Corollary 4.1.8]) and R is noetherian, this implies it is projective. $\square$

Corollary 4.3.2. *If $\kappa : T_0/\overline{Z(K)} \to R^\times$ is a weight and F_κ has a slope $\leq h$-factorization, then $S_\kappa^{<r,\circ}(K)_{\leq h}$ and $S_{\kappa,\xi}^{<r,\circ}(K)_{\leq h}$ are finite R_0-modules.*

Proof. This follows from the equality $H^0(K, \mathcal{D}_\kappa^r)_{\leq h} = H^0(K, \mathcal{D}_\kappa^{<r})_{\leq h}$, and the fact that $\mathcal{D}_\kappa^{<r,\circ}$ is bounded in $\mathcal{D}_\kappa^{<r}$. $\square$

Now we consider the behavior of $H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h}$ under change of coefficients. Let $\kappa_R : T_0/\overline{Z(K)} \to R^\times$ be a weight. If $f : R \to R'$ is a homomorphism of pseudoaffinoid algebras, we let $\kappa_{R'}$ denote the composition $T_0/\overline{Z(K)} \xrightarrow{\kappa_R} R^\times \xrightarrow{f} R'^\times$. By [JN16, Corollary A.14], f is topologically of finite type, so we have a surjection $R\langle X_1, \ldots, X_n\rangle \twoheadrightarrow R'$. If $R_0 \subset R$ is a ring of definition and $u \in R_0$ is a pseudo-uniformizer, we define $R_0' := R_0\langle X_1, \ldots, X_n\rangle$ and $u' := f(u)$.

We extend $|\cdot|_R$ to $R\langle X_1, \ldots, X_n\rangle$ via

$$|\sum_\alpha r_\alpha \underline{X}^\alpha|_R := \sup_\alpha |r_\alpha|_R$$

where α is a multi-index, and we equip R' with the quotient norm $|\cdot|_{R'}$. It follows from the discreteness of $\log_p|\cdot|_R$ that the unit ball of R' is R_0'.

Lemma 4.3.3. *With notation as above, the natural map $R_0' \widehat{\otimes}_{R_0} \mathcal{D}_{\kappa_R}^{<r,\circ} \to \mathcal{D}_{\kappa_{R'}}^{<r,\circ}$ is a topological isomorphism (with respect to the u'-adic topology), where the completed tensor product is taken with respect to the u-adic topology on $\mathcal{D}_{\kappa_R}^{<r,\circ}$ and the u'-adic topology on R_0'.*

Proof. Since f factors as $R \hookrightarrow R\langle X_1, \ldots, X_n\rangle \twoheadrightarrow R'$, and the result is clear for $R_0 \to R_0\langle X_1, \ldots, X_n\rangle$, we may assume that f is surjective.

We first check that the morphism $R_0' \widehat{\otimes}_{R_0} \mathcal{D}_{\kappa_R}^{<r,\circ} \to \mathcal{D}_{\kappa_{R'}}^{<r,\circ}$ is an isomorphism of R_0'-modules. The discussion after [JN16, Proposition 3.2.7] shows that

$$\mathcal{D}_{\kappa_R}^{<r,\circ} \cong \prod_{\alpha} R_0 \cdot u^{-n_R(r,u,\alpha)} \mathbf{n}^{\alpha}$$

where $n_R(r,u,\alpha) := \left\lfloor \frac{|\alpha| \log_p r}{\log_p |u|_R} \right\rfloor$, $\mathbf{n}$ is a certain (non-canonical but explicit) finite set (depending only on the group-theoretic data we fixed at the beginning of §3), and α is a multi-index (and similarly for $\mathcal{D}_{\kappa_{R'}}^{<r,\circ}$). Now R_0' is a finitely presented R_0-module, and for any finitely presented R_0-module M, the natural morphism $M \otimes_{R_0} \prod_\alpha R_0 \cdot u^{-n_R(r,u,\alpha)} \mathbf{b}^\alpha \to \prod_\alpha M \cdot u^{-n_R(r,u,\alpha)} \mathbf{b}^\alpha$ is an isomorphism. By construction, $n_R(r,u,\alpha) = n_{R'}(r,u',\alpha)$ for all α, so the claim follows.

Finally, the morphism $R_0' \widehat{\otimes}_{R_0} \mathcal{D}_{\kappa_R}^{<r,\circ} \to \mathcal{D}_{\kappa_{R'}}^{<r,\circ}$ is clearly continuous, so the open mapping theorem implies that it is a topological isomorphism. $\square$

Corollary 4.3.4. *Suppose $R \to R'$ is a morphism of pseudoaffinoid algebras, and suppose $R_0' \subset R'$ is a ring of definition containing the image of R_0. Then there is a topological isomorphism $R_0' \otimes_{R_0} S_{\kappa,\xi}^{<r,\circ}(K)_{\leq h} \xrightarrow{S} {}_{\kappa',\xi}^{<r,\circ}(K)_{\leq h}$*

Proof. We define

$$H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{>h} := \operatorname{im}\left(H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi] \to H^0(K, \mathcal{D}_\kappa^r)[\xi]_{>h}\right)$$

so that $H^0(K, \mathcal{D}_\kappa^{<r,\circ}) \cong H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h} \oplus H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{>h}$.

Writing $D^\times \backslash (\mathbf{A}_{F,f} \otimes_F D)^\times / K = \coprod_{i \in I} D^\times g_i K$ for some finite set of elements $g_i \in (\mathbf{A}_{F,f} \otimes_F D)^\times$, we have an isomorphism $H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi] \cong \oplus_{i\in I} \left(\mathcal{D}_\kappa^{<r,\circ}\right)^{(K\mathbf{A}_{F,f}^\times \cap g_i^{-1} D^\times g_i)/F^\times}$. For every quotient map $R \to R'$, Lemma 4.3.3 implies that the base change map

$$R_0' \otimes_{R_0} \oplus_i \mathcal{D}_\kappa^{<r,\circ} \to \oplus_i \mathcal{D}_{\kappa'}^{<r,\circ}$$

is an isomorphism. Moreover, the calculations of [Tay06, Lemma 1.1] show that the order of $(K\mathbf{A}_{F,f}^\times \cap g_i^{-1} D^\times g_i)/F^\times$ is prime to p for all i, so the base change map $R_0' \otimes_{R_0} H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi] \to H^0(K, \mathcal{D}_{\kappa'}^{<r,\circ})[\xi]$ is an isomorphism.

On the other hand, base change carries $H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h}$ to $H^0(K, \mathcal{D}_{\kappa'}^{<r,\circ})[\xi]_{\leq h}$ and $H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{>h}$ to $H^0(K, \mathcal{D}_{\kappa'}^{<r,\circ})[\xi]_{>h}$. Since $R_0' \otimes_{R_0} H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi] \to H^0(K, \mathcal{D}_{\kappa'}^{<r,\circ})[\xi]$ is an isomorphism, this implies that $R_0' \otimes_{R_0} H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h} \to H^0(K, \mathcal{D}_{\kappa'}^{<r,\circ})[\xi]_{\leq h}$ is also an isomorphism. $\square$

Corollary 4.3.5. *Suppose R is reduced. Let $\kappa : T_0/\overline{Z(K)} \to R^\times$ be a weight. If F_κ has a slope-$\leq h$ factorization, then $H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h}$ is a finite projective R_0-module.*

Proof. We may assume $\operatorname{Spa} R$ is connected, so that $H^0(K, \mathcal{D}_\kappa^{<r})[\xi]_{\leq h}$ is a finite projective R-module of some constant rank d.

To prove that $H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h}$ is a finite projective R_0-module, we proceed by Noetherian induction on $\operatorname{Spec} R_0$. If R is a local field, this follows because $H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h}$ is a lattice (hence u-torsion-free) in the finite-dimensional R-vector space $H^0(K, \mathcal{D}_\kappa^{<r})[\xi]_{\leq h}$.

For general R, it suffices by [Sta18, Tag 0FWG] to show that $k(\mathfrak{p}) \otimes_{R_0} H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h}$ has constant rank d, where $k(\mathfrak{p})$ is the residue field as $\mathfrak{p}$ varies over $\operatorname{Spec} R_0$. This holds whenever $\mathfrak{p} \in \operatorname{Spec} R_0 \smallsetminus V(u) = \operatorname{Spec} R$ because $H^0(K, \mathcal{D}_\kappa^{<r})_{\leq h}$ is a finite projective R-module of rank d.

Since R_0 has no u-torsion, the remaining prime ideals of R_0 have the form $\sqrt{\mathfrak{q} + (u)}$, where $\mathfrak{q} = \mathfrak{p} \cap R_0$ for some $\mathfrak{p} \in \operatorname{Spec} R$. Setting $R_0' := R_0/\mathfrak{q}$, the inductive hypothesis implies that $R_0' \otimes_{R_0} H^0(K, \mathcal{D}_\kappa^{<r,\circ})[\xi]_{\leq h}$ is finite projective of rank d, so we are done. □

In fact, we can prove something stronger:

Proposition 4.3.6. *Suppose R is reduced. For any finite set of primes Q as in section 4.1, the module $S_{\kappa,\xi}^{<r,\circ}(K^-(Q))_{\leq h}$ is finite projective over $R_0[\Delta_Q]$ and the natural map*

$$\sum_{h \in \Delta_Q} \langle h \rangle : \left(S_{\kappa,\xi}^{<r,\circ}(K^-(Q))_{\leq h} \right)_{\Delta_Q} \to S_{\kappa,\xi}^{<r,\circ}(K_0(Q))_{\leq h}$$

is an isomorphism.

Proof. We first assume that K is neat. Then we can write $\mathbf{A}_{F,f} \otimes_F D^\times = \coprod_i D^\times g_i K$, where the disjoint union is finite, and $K_Q(0) = \coprod_{j \in \Delta_Q} k_j K^-(Q)$. We claim that Δ_Q acts freely on $D^\times \backslash (\mathbf{A}_{F,f} \otimes_F D)^\times / K^-(Q)$, from which the result follows. But if $D^\times g_i k_j K^-(Q) = D^\times g_{i'} k_{j'} K^-(Q)$, then the neatness hypothesis 3.1.1 implies that $i = i'$ and $j = j'$, and the result follows.

If $K' \lhd K$ with K' neat and $[K : K']$ prime to p, then $S_{\kappa,\xi}^{<r,\circ}(K^-(Q))_{\leq h} = \left(S_{\kappa,\xi}^{<r,\circ}(K_Q')_{\leq h} \right)^{K/K'}$. Since $S_{\kappa,\xi}^{<r,\circ}(K_Q')_{\leq h}$ is projective, its invariants by a prime-to-p group are also projective. □

In addition to Lemma 4.2.1, we can extend [Kis09a, Lemma 2.1.7] to a statement about integral families of overconvergent modular forms.

Proposition 4.3.7. *The map*

$$\prod_{v \in Q} (U_{\varpi_v} - B_v) : S_{\kappa,\xi}^{<r,\circ}(K)_{\leq h, \mathfrak{m}} \to S_{\kappa,\xi}^{<r,\circ}(K_0(Q))_{\leq h, \mathfrak{m}_{Q,0}}$$

is an isomorphism (where we view $S_{\kappa,\xi}^{<r,\circ}(K)_{\leq h,\mathfrak{m}}$ as a submodule of $S_{\kappa,\xi}^{<r,\circ}(K_0(Q))_{\leq h,\mathfrak{m}}$).

Proof. We may assume $Q = \{v\}$, by induction on the size of Q. Then the source and the target are finite projective R_0-modules, and by [Kis09a, Lemma 2.1.7] the map is an isomorphism when specialized to any sufficiently large classical weight. It follows that $S^{<r,\circ}_{\kappa,\xi}(K)_{\leq h,\mathfrak{m}}$ and $S^{<r,\circ}_{\kappa,\xi}(K_0(Q))_{\leq h,\mathfrak{m}_{Q,0}}$ have the same rank, so it suffices to check that $U_{\varpi_v} - B_v$ is surjective after specializing at every maximal ideal of R_0.

Thus, we need to check that

$$U_{\varpi_v} - B_v : \mathbf{F}' \otimes_{R_0} S^{<r,\circ}_{\kappa,\xi}(K)_{\leq h,\mathfrak{m}} \to \mathbf{F}' \otimes_{R_0} S^{<r,\circ}_{\kappa,\xi}(K_0(Q))_{\leq h,\mathfrak{m}_{Q,0}}$$

is surjective for any specialization $R_0 \to \mathbf{F}'$ at a maximal ideal. But this is a map of vector spaces of the same dimension, so it is enough to prove injectivity.

The module $\mathbf{F}' \otimes_{R_0} \left(S^{<r,\circ}_{\kappa,\xi}(K)_{\leq h}\right)_{\mathfrak{p}}$ is a finite module over the artin local ring $\mathbb{T}_{\mathfrak{m}}/p$, so if the kernel of $U_{\varpi_v} - B_v$ is non-trivial, it contains $f \neq 0$ which is $\mathfrak{m}$-torsion. In particular, $T_v(f) = (\alpha_v + \beta_v)x$ and $U_{\varpi_v}(f) = \beta_v$.

Since

$$[K_0(v) \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right) K_0(v)] = \coprod_{\alpha \in k_v} \left(\begin{smallmatrix} 1 & \\ \widetilde{\alpha}\varpi_v & \varpi_v \end{smallmatrix}\right) K_0(v)$$

and

$$[\mathrm{GL}_2(\mathscr{O}_{F_v}) \left(\begin{smallmatrix} 1 & \\ & \varpi_v \end{smallmatrix}\right) \mathrm{GL}_2(\mathscr{O}_{F_v})] = \left(\begin{smallmatrix} \varpi_v & \\ & 1 \end{smallmatrix}\right) \mathrm{GL}_2(\mathscr{O}_{F_v}) \bigsqcup \coprod_{\alpha \in k_v} \left(\begin{smallmatrix} 1 & \\ \widetilde{\alpha}\varpi_v & \varpi_v \end{smallmatrix}\right) \mathrm{GL}_2(\mathscr{O}_{F_v})$$

where $\widetilde{\alpha}$ denotes a lift of α, we see that

$$\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right) | f = (T_v - U_{\varpi_v})(f) = \alpha_v f$$

Then

$$\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right) | f = \left(\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right)\right) | f = \left(\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right) | f = \alpha_v f$$

(since f is fixed by the action of $\mathrm{GL}_2(\mathscr{O}_{F_v})$). It follows that

$$U_{\varpi_v}(f) = \sum_{\alpha \in k_v} \left(\begin{smallmatrix} 1 & 0 \\ \widetilde{\alpha}\varpi_v & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & \\ & \varpi \end{smallmatrix}\right) | f = |k_v| \alpha_v f = \alpha_v f$$

which contradicts the assumption that $\alpha_v \neq \beta_v$. □

5. Patching and modularity

5.1. **Set-up.** Let us recall our goal. Fix a non-archimedean local field L with ring of integers $\mathscr{O}_L$, residue field $\mathbf{F}_q$, and uniformizer u. Fix a continuous odd representation $\overline{\rho} : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{F}_q)$, such that:

- $\overline{\rho}$ is modular
- $\overline{\rho}|_{\mathrm{Gal}_{\mathbf{Q}(\zeta_p)}}$ is absolutely irreducible
- The image of $\overline{\rho}$ contains $\mathrm{SL}_2(\mathbf{F}_p)$
- $\overline{\rho}$ is unramified at all places away from p
- $\overline{\rho} \not\sim \chi \otimes \begin{pmatrix} \overline{\chi}_{\mathrm{cyc}} & * \\ & 1 \end{pmatrix}$ for any character $\chi : \mathrm{Gal}_{\mathbf{Q}} \to \mathbf{F}_q^{\times}$.

The assumption that $\overline{\rho}$ has large image is stronger than the typical hypothesis. This is because we need to use [BH17, Theorem B.0.1] to ensure that we can work with middle-degree eigenvarieties for Hilbert modular forms.

We wish to prove the following modularity theorem:

Theorem 5.1.1. *Suppose $\rho : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\mathscr{O}_L)$ is a continuous odd representation unramified away from p and trianguline at p with regular parameters, whose reduction modulo u is as above. Then ρ is the twist of a Galois representation arising from an overconvergent modular form.*

The predicted weight κ can be read off from the parameters of the triangulation, as can the predicted slope h.

More precisely, we will show that ρ corresponds to a class in $S_\kappa(K)_{\leq h}$, where $K = I \cdot K_1(N)^p = I \cdot \prod_{\ell \neq p, \ell \nmid N} \mathrm{GL}_2(\mathbf{Q}_\ell) \cdot \prod_{\ell | N} K_1(\ell)$ for some $N \geq 5$ prime to p. To do this, we will consider an open weight $\kappa : T_0 \to \mathscr{O}(U)^\times$, where $U \subset \mathscr{W}$ contains a point corresponding to κ and (U, h) is a slope datum, and we will study the spaces $S_\kappa(K^-(Q))_{\leq h}$ for varying sets of primes Q.

5.2. **Patched eigenvarieties.** In this section, we construct patched quaternionic eigenvarieties, using the language of ultrafilters of [Sch18, §9]. We fix a totally real field F split at all places above p and a totally definite quaternion algebra D over F, which is ramified at all infinite places and split at all finite places. We also fix the tame level $K^p := \mathrm{GL}_2(\mathbf{A}_{F,f}^p)$. We further assume that $F/\mathbf{Q}$ is abelian, so that Leopoldt's conjecture is known to hold. Unlike [Sch18], we do not assume that F has a unique prime above p; we let $\Sigma_p := \{v \mid p\}$. We expect these hypotheses can be relaxed considerably, but this is not necessary for our applications. Fix some finite extension $E/\mathbf{Q}_p$ with residue field containing $\mathbf{F}_q$.

Recall that there are Galois deformation rings $R_{\overline{\rho},\Sigma_p}^{\square}$ and $R_{\overline{\rho},\Sigma_p}$, parametrizing deformations of $\overline{\rho}$ unramified outside of Σ_p, where $R_{\overline{\rho}}^{\square}$ additionally parametrizes framings of the deformations at places of Σ_p. There is also a local framed deformation ring $R_{\overline{\rho},\mathrm{loc}}^{\square} := \widehat{\bigotimes}_{v \in \Sigma_p} R_{\overline{\rho}_v}^{\square}$, where $R_{\overline{\rho}_v}^{\square}$ parametrizes framed deformations of $\overline{\rho}|_{\mathrm{Gal}_{F_v}}$, and there is a natural map $R_{\overline{\rho},\mathrm{loc}}^{\square} \to R_{\overline{\rho}}^{\square}$.

We define a distinguished family of characters $\eta_{\mathrm{univ}} : \mathrm{Gal}_F \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$ over integral weight space. We have a universal weight $\underline{\lambda} = (\lambda_1, \lambda_2)$, where each λ_i is a character $\prod_{v \in \Sigma_p} \mathscr{O}_{F_v}^\times \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$, and we define $\eta_v : \mathscr{O}_{F_v}^\times \cong \mathbf{Z}_p^\times \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$ via $\eta(x) := \left(\lambda_1|_{\mathscr{O}_{F_v}^\times}(x)\lambda_2|_{\mathscr{O}_{F_v}^\times}(x)\right)^{-1}$. Then because we have assumed that Leopoldt's conjecture holds for F, we see that η_v is independent of $v \in \Sigma$; global class field theory gives us a corresponding character $\mathrm{Gal}_{\mathbf{Q}} \to \mathbf{Z}_p[\![T_0/\overline{Z(K)}]\!]^\times$, which we restrict to Gal_F to obtain η_{univ}.

We fix an unramified continuous character $\psi_0 : \mathrm{Gal}_F \to \mathscr{O}_E[\![T_0/\overline{Z(K)}]\!]^\times$ such that the reduction $\overline{\psi}_0$ modulo the maximal ideal satisfies $\det \overline{\rho} = \overline{\psi}_0 \overline{\eta}_{\mathrm{univ}} \overline{\chi}_{\mathrm{cyc}}^{-1}$, and we set $\psi := \psi_0\eta_{\mathrm{univ}}$ and $\psi' := \psi_0\eta_{\mathrm{univ}}\chi_{\mathrm{cyc}}^{-1}$. Then we constructed quotients

$$\mathscr{O}_E[\![T_0/\overline{Z(K)}]\!] \mathbin{\widehat{\otimes}} R^{\square}_{\overline{\rho},\Sigma_p} \twoheadrightarrow R^{\square,\psi'}_{\overline{\rho},\Sigma_p}$$
$$\mathscr{O}_E[\![T_0/\overline{Z(K)}]\!] \mathbin{\widehat{\otimes}} R^{\square}_{\overline{\rho},\mathrm{loc}} \twoheadrightarrow R^{\square,\psi'}_{\overline{\rho},\mathrm{loc}}$$
$$\mathscr{O}_E[\![T_0/\overline{Z(K)}]\!] \mathbin{\widehat{\otimes}} R_{\overline{\rho},\Sigma_p} \twoheadrightarrow R^{\psi'}_{\overline{\rho},\Sigma_p}$$

parametrizing families of deformations with fixed determinants.

We also define families of weights $\underline{\kappa}_v$ over $\mathscr{W}_F$ via

$$\underline{\kappa}_v = (\kappa_{v_1}, \kappa_{v,2}) = \left(\lambda_2|^{-1}_{\mathscr{O}^\times_{F_v}}, \lambda_1|^{-1}_{\mathscr{O}^\times_{F_v}} \chi^{-1}_{\mathrm{cyc}}\right)$$

In order to find sets of Taylor–Wiles primes, we impose the following standard hypotheses:

(1) $p \geq 5$
(2) $\overline{\rho}|_{F(\zeta_p)}$ is absolutely irreducible
(3) If $p = 5$ and $\overline{\rho}$ has projective image $\mathrm{PGL}_2(\mathbf{F}_5)$, the kernel of $\overline{\rho}$ does not fix $F(\zeta_5)$

Then we have the following relative version of [Kis09a, Proposition 2.2.4] (since we assumed p splits completely in F, $[F : \mathbf{Q}] = |\Sigma_p|$):

Proposition 5.2.1. *Let $g := \dim_{\mathbf{F}_q} H^1(\mathrm{Gal}_{F,\Sigma_p}, \mathrm{ad}^0\,\overline{\rho}(1)) - 1$. Then for each positive integer n, there exists a finite set Q_n of places of F, disjoint from Σ_p, of cardinality $g + 1$, such that*

(1) *for all $v \in Q_n$, $\mathrm{Nm}(v) \equiv 1 \pmod{p^n}$, and $\overline{\rho}(\mathrm{Frob_v})$ has distinct eigenvalues*
(2) *the global relative Galois deformation ring $R^{\square,\psi'}_{\overline{\rho},\Sigma_p\cup Q_n}$ parametrizing families of deformations with determinant ψ unramified outside $\Sigma_p \cup Q_n$ can be topologically generated as an $R^{\square,\psi'}_{\overline{\rho},\mathrm{loc}}$-algebra by g elements.*

Proof. This follows from Lemma 2.1.1, as in [Kis09b, Proposition 3.2.5]. □

We fix such a set Q_n for each $n \geq 1$, as well as a non-principal ultrafilter $\mathfrak{F}$ on $\{n \geq 1\}$ (more precisely, on its power set, ordered by inclusion). For notational convenience, we set $Q_0 := \emptyset$, and we let $Q'_n := Q_n \cup \Sigma_p$. For each n, we again let $K^-(Q_n) \subset K_0(Q_n) \subset G(\mathbf{A}^p_{F,f}) \cong \mathrm{GL}_2(\mathbf{A}^p_{F,f})$ be the compact open subgroups

$$K^-(Q_n) := \prod_{v\notin Q_n} \mathrm{GL}_2(\mathscr{O}_{F_v}) \times \prod_{v\in Q_n} K^-(v) \subset \prod_{v\notin Q_n} \mathrm{GL}_2(\mathscr{O}_{F_v}) \times \prod_{v\in Q_n} K_0(v)$$

Let $\xi : \mathbf{A}^\times_{F,f}/F^\times \to \mathscr{O}(\mathscr{W}_F)^\times$ be the central character corresponding to ψ via class field theory. Now we fix a weight $\kappa : T_0 \to \mathscr{O}(U)^\times$, $U = \mathrm{Spa}\, R$, such that (U, h) is a slope datum for $S_{\kappa,\xi}(K)$ (by Corollary 4.2.2 and Lemma 4.2.3,

(U,h) is also a slope datum for $S_{\kappa,\xi}(K_0(Q_n))$ and $S_{\kappa,\xi}(K^-(Q_n))$ for all n). We equip R with a norm adapted to κ, using Lemma 4.1.1, and we let $R_0 \subset R$ be the unit ball. Let $Z_{U,h}$ denote the corresponding piece of the spectral variety $\mathscr{Z}$ for $S_{\kappa,\xi}(K)$. Then we fix some $r > r_\kappa$.

The modularity of the residual representation $\overline{\rho}$ means that $\overline{\rho}$ corresponds to a maximal ideal $\mathfrak{m} \subset \mathbb{T}$. For each $v \in Q_n$, we fix a root α_v of the characteristic polynomial $X^2 - T_v X + \mathrm{Nm}(v) S_v$ of $\overline{\rho}(\mathrm{Frob_v})$ (increasing $\mathbf{F}_q$, and hence E, if necessary), and we consider the corresponding maximal ideal $\mathfrak{m}_{Q_n} \subset \mathbb{T}^-_{Q_n}$ (as in § 3.6. Then we have a collection of diagrams

$$\begin{tikzcd} \mathscr{O}_E \times \mathscr{X}^{K^-(Q_n)}_{\underline{D}^\times/F,\xi,\mathfrak{m}_{Q_n}} \arrow[r] \arrow[d, "\mathrm{wt}"] & \coprod_{\overline{\rho}} \mathrm{Spa}\, R_{\overline{\rho},Q_n\cup\Sigma_p} \\ \mathscr{O}_E \times \mathscr{W}_F & \end{tikzcd}$$

The pre-image $\mathrm{wt}^{-1}(U)$ has the form $\mathrm{Spa}\left(\mathbb{T}_{K^-(Q_n),\kappa,\xi,\le h}, \mathbb{T}^\circ_{K^-(Q_n),\kappa,\xi,\le h}\right)$, and since $\mathscr{X}^{K^-(Q_n)}_{\underline{D}^\times/F,\xi,\mathfrak{m}_{Q_n}}$ is reduced, $\mathbb{T}^\circ_{K^-(Q_n),\kappa,\xi,\le h} \subset \mathbb{T}_{K^-(Q_n),\kappa,\xi,\le h}$ is a ring of definition. Another ring of definition is provided by $\mathbb{T}^{<r,\circ}_{K^-(Q_n),\kappa,\xi,\le h}$.

For each n, the module of overconvergent modular forms $S_{\kappa,\xi}(K^-(Q_n))$ is a $\mathbb{T}_{K^-(Q_n),\kappa,\xi,\le h}$-module. The $\mathbb{T}^{<r,\circ}_{K^-(Q_n),\kappa,\xi,\le h}$-submodule $S^{<r,\circ}_{\kappa,\xi}(K^-(Q_n))$ is a lattice in $S_{\kappa,\xi}(K^-(Q_n))$.

Let $R^{\psi'}_{\overline{\rho},Q_n'}|_U$ denote the ring of definition $R_0 \widehat{\otimes}_{\mathscr{O}_E[\![T_0/\overline{Z(K)}]\!]} R^{\psi'}_{\overline{\rho},Q_n'}$ of $U \times \mathrm{Spa}\, R^{\psi'}_{\overline{\rho},Q_n'}$. Using the existence of Galois representations, we see that $\mathbb{T}^{<r,\circ}_{K^-(Q_n),\kappa,\xi,\le h}$ is a $R^{\psi'}_{\overline{\rho},Q_n'}|_U$-algebra.

By Lemma 4.3.6 $S^{<r,\circ}_{\kappa,\xi}(K^-(Q_n))_{\le h,\mathfrak{m}^-_{Q_n}}$ is a projective $R_0[\Delta_{Q_n}]$-module, with $R_0 \otimes_{R_0[\Delta_{Q_n}]} S^{<r,\circ}_{\kappa,\xi}(K^-(Q_n))_{\le h,\mathfrak{m}^-_{Q_n}} \cong S^{<r,\circ}_{\kappa,\xi}(K_0(Q_n))_{\le h,\mathfrak{m}_{0,Q_n}}$. Set $j = 4|\Sigma_p| - 1$ and $k = |Q_n| = g+1$. Using local-global compatibility at places in Q_n, there is a homomorphism $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_k]\!] \to R^{\psi'}_{\overline{\rho},Q_n'}|_U$ such that the action of $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_k]\!]$ on $S^{<r,\circ}_{\kappa,\xi}(K^-(Q_n))^\circ_{\le h,\mathfrak{m}^-_{Q_n}}$ is compatible with the action of $R_0[\Delta_{Q_n}]$ via a fixed surjection $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\ldots,y_k]\!] \to R_0[\Delta_{Q_n}]$.

We observe that we may view $S_{\kappa,\xi}(K^-(Q)n))_{\le h,\mathfrak{m}^-_{Q_n}}$ as a module over $\mathrm{Spa}\, R[\Delta_Q] \times_U \mathbb{B}_{U,h}$, where $\mathbb{B}_{U,h}$ is the ball of radius h, by letting the coordinate on $\mathbb{B}_{U,h}$ act as U_p^{-1}. Then the support of $S_{\kappa,\xi}(K^-(Q)n))_{\le h,\mathfrak{m}^-_{Q_n}}$ is a nilpotent thickening of $Z_{U,h}$.

Moreover, we may use local-global compatibility at places in Σ_p to make $S_{\kappa,\xi}(K^-(Q_n))^\circ_{\leq h,\mathfrak{m}^-_{Q_n}}$ into a module over $R^{\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},Q'_n,\leq h}|_U$, where the coordinates of $\mathbf{G}_m^{\Sigma_p}$ act as U_v^{-1}.

Since $R^{\psi'}_{\overline{\rho},Q'_n} \to R^{\square,\psi'}_{\overline{\rho},Q'_n}$ is formally smooth of dimension j, we may construct a homomorphism

$$R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_1,\ldots,y_k,y_{k+1},\ldots,y_{k+j}]\!] \to R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},Q'_n,\leq h}|_U$$

compatible with

$$R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_1,\ldots,y_k]\!] \to R^{\psi'}_{\overline{\rho},Q'_n}|_U$$

such that $y_{k+1},\ldots,y_{k+j}$ are the framing variables. Finally, we fix a surjection $R^{\square,\psi'}_{\overline{\rho},\mathrm{loc}}[\![x_1,\ldots,x_g]\!] \twoheadrightarrow R^{\square,\psi'}_{\overline{\rho},Q'_n}$ and a map $R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_1,\ldots,y_{k+j}]\!] \to R^{\square,\psi'}_{\overline{\rho},\mathrm{loc}}[\![x_1,\ldots,x_g]\!]$ such that the corresponding diagram

$$\begin{tikzcd}
R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_1,\ldots,y_{k+j}]\!] \arrow[r] \arrow[dr] & R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}[\![x_1,\ldots,x_g]\!]|_U \arrow[d] \\
& R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},Q'_n,\leq h}|_U
\end{tikzcd}$$

commutes.

Now we can patch. Set $M_n := R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},Q'_n,\leq h} \otimes_{R^{\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},Q'_n,\leq h}} S^{<r,\circ}_{\kappa,\xi}(K^-_{Q_n})_{\leq h,\mathfrak{m}^-_{Q_n}}$, so that $R_0 \otimes_{R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_1,\ldots,y_{k+j}]\!]} M_n \cong S^{<r,\circ}_{\kappa,\xi}(K)_{\leq h,\mathfrak{m}}$ for all $n \geq 1$.

For any open ideal $I \subset R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]$, the quotient $M_n/(I+(y_i))$ is the reduction of the projective R_0-module $S^{<r,\circ}_{\kappa,\xi}(K)_{\leq h,\mathfrak{m}}$; this implies that M_n/I is projective over $R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]/I$, with the rank fixed and the number of generators and relations bounded uniformly in n. We may take an ultraproduct as in [Sch18, §8] and conclude that $\prod_{n\geq 1} M_n/I$ is finitely presented over $\prod_{n\geq 1} R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]/I$. For any such I, our choice of non-principal ultrafilter gives a localization map

$$\prod_{n\geq 1} R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]/I \to R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]/I$$

and hence we may define

$$M_I := R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]/I \otimes_{\prod_{n\geq 1} R_0 \mathbin{\widehat{\otimes}} \mathbf{Z}_p[\![y_i]\!]/I} \prod_n M_n/I$$

Passing to the inverse limit, we obtain the patched module

$$M_\infty := \varprojlim_I M_I$$

Similarly, we may define patched global deformation rings $R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty,\leq h,I}|_U$ and $\mathbb{T}^{<r,\circ}_{\infty,\kappa,\leq h,I}$ via

$$R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty,\leq h,I}|_U := R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I \otimes_{\prod_{n\geq 1} R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I} \prod_n R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},Q'_n,\leq h}|_U/I$$

and

$$\mathbb{T}^{<r,\circ}_{\infty,\kappa,\leq h,I} := R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I \otimes_{\prod_{n\geq 1} R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I} \prod_n \mathbb{T}^{<r,\circ}_{K^-(Q_n),\kappa,\leq h}/I$$

Setting $R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U := \varprojlim_I R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty,\leq h,I}|_U$ and $\mathbb{T}^{<r,\circ}_{\infty,\kappa,\leq h} := \varprojlim_I \mathbb{T}^{<r,\circ}_{\infty,\kappa,\leq h,I}$, we have a sequence of homomorphisms

$$R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!] \to R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}[\![x_i]\!]|_U \to R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U \to \mathbb{T}^{<r,\circ}_{\infty,\kappa,\leq h}$$

compatible with their actions on M_∞.

Proposition 5.2.2. *M_∞ is a finite projective $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_1,\dots,y_{k+j}]\!]$-module.*

Proof. The powers of the ideal $(u, y_1,\dots,y_{k+j})$ are cofinal in the set of open ideals of $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]$, and for any open ideals $I \subset I' \subset R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]$, the natural map $M_I/I' \to M_{I'}$ is an isomorphism. Then [Sta18, Tag 09B8] implies that M_∞ is complete and $M_\infty/I \cong M_I$ for all open ideals $I \subset R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]$, and [Mat89, Theorem 8.4] then implies that M_∞ is finite.

For any homomorphism $R_0 \to R'_0$ and any open ideal $I \subset R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]$, the natural morphism $R'_0 \otimes_{R_0} \prod_n R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I \to R'_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I$ factors as

$$R'_0 \otimes_{R_0} \prod_n R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I \to \prod_n R'_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I \to R'_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I$$

Thus, if $R_0 \to R'_0$ is a topologically finite type morphism such that $R'_0 \otimes_{R_0} M_n$ is free of rank d for all n, we see that $R'_0 \otimes_{R_0} M_I$ is free of rank d, as well. We may find such a morphism by choosing a cover of $\mathrm{Spa}\, R_0$ trivializing $S_{\kappa,\xi}(K)^\circ_{\leq h,\mathfrak{m}}$. Since M_∞ is $(u, y_1,\dots,y_{k+j})$-adically separated, this combined with [Mat89, Theorem 22.3] implies that M_∞ is flat over $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]$, and hence projective. $\square$

Lemma 5.2.3. *Let $f : R_0 \to R'_0$ be a finite morphism, where R'_0 is a noetherian ring of definition in a pseudoaffinoid algebra. Let κ' be the weight $f \circ \kappa$, and let M'_∞ denote the patched module constructed from the modules of modular forms $S^{<r,\circ}_{\kappa',\xi}(K^-_{Q_n})_{\leq h,\mathfrak{m}^-_{Q_n}}$. Then there is an isomorphism*

$$R'_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!] \otimes_{R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]} M_\infty \xrightarrow{\sim} M'_\infty$$

Proof. Let $M'_n := R^{\square,\psi',\underline{\kappa}'}_{\mathrm{tri},\overline{\rho},Q'_n,\leq h} \otimes_{R^{\psi',\underline{\kappa}'}_{\mathrm{tri},\overline{\rho},Q'_n,\leq h}} S^{<r,\circ}_{\kappa',\xi}(K^-_{Q_n})_{\leq h,\mathfrak{m}^-_{Q_n}}$. The open ideals $I \subset R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]$ generate open ideals of $R'_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]$ and are cofinal,

so it suffices to show that we have an isomorphism

$$R_0' \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I \otimes_{R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I} M_I \xrightarrow{\sim} M_I' := \left(R_0' \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I\right) \otimes_{\prod_{n\geq 1} R_0' \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I} \prod_{n\geq 1} M_n'/I$$

The left side is isomorphic to $R_0' \otimes_{R_0} M_I$ (because $R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]/I$ is discrete, by construction). We may choose a presentation $R_0^{\oplus n_1} \to R_0^{\oplus n_2} \to R_0'$; because the transition maps $\prod_{n=1}^{k+1} M_n \to \prod_{n=1}^{k} M_n$ are surjective, the Mittag-Leffler condition implies that the natural map

$$R_0' \otimes_{R_0} M_I \to M_I'$$

is an isomorphism. $\square$

In particular, we see that the formation of M_∞ is compatible with change in the ring of definition of R, and with passage to closed subspaces of U. We remark that the same argument shows that the formation of $R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U$ and $\mathbb{T}^{<r,\circ}_{\infty,\kappa,\leq h}$ are compatible with change in the ring of definition of R.

Now we pass to the loci of the corresponding map

$$\mathrm{Spa}\, R^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}[\![x_i]\!]|_U \to \mathrm{Spa}\, R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]$$

where $u \neq 0$, and we consider the analytification M_∞^{an} of M_∞ as a coherent sheaf over $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}|_U \times \mathrm{Spa}\, \mathbf{Z}_p[\![x_i]\!]$.

The support of M_∞^{an} over $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}|_U \times \mathrm{Spa}\, \mathbf{Z}_p[\![x_i]\!]$ is a Zariski-closed subspace; since M_∞^{an} is a vector bundle over $\mathrm{Spa}\, R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]^{\mathrm{an}}$, the dimension of its support must be equal to

$$\dim \mathrm{Spa}\, R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]\left[\frac{1}{u}\right] = \dim U + (g+1) + (4|\Sigma_p| - 1) = \dim U + g + 4|\Sigma_p|$$

But the morphism $X^{\square,\psi',\underline{\kappa}_n}_{\mathrm{tri},\overline{\rho},\mathrm{loc}}|_U \to U$ has relative dimension $4|\Sigma_p|$ over an open subspace of U by Proposition 2.3.5, so any non-empty irreducible components have total dimension $\dim U + 4|\Sigma_p|$. It follows that the support of M_∞^{an} on $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}|_U \times \mathrm{Spa}\, \mathbf{Z}_p[\![x_i]\!]$ is the union of irreducible components.

Finally, since we have a closed embedding

$$X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U \hookrightarrow X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc},\leq h}|_U \times \mathrm{Spa}\, \mathbf{Z}_p[\![x_i]\!]$$

we conclude that the support of M_∞ on $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty,\leq h}|_U$ is also a union of irreducible components, which we denote $\mathscr{X}^{\infty,\psi,\kappa,h}_{\underline{D}^\times,U}$.

Now since $U_p^{-1} = \prod_{v|p} U_v^{-1}$ acts on M_∞^{an} by construction, we may also view M_∞^{an} as a coherent sheaf on $\mathrm{Spa}\, R_0 \widehat{\otimes} \mathbf{Z}_p[\![y_i]\!]\left[\frac{1}{u}\right] \times_U \mathbb{B}_{U,h}$. Since the support of each M_n/I is a nilpotent thickening of $Z_{U,h}$, we see that the support of M_∞^{an} is a closed subspace of $Z_{U,h} \times \mathrm{Spa}\, \mathbf{Z}_p[\![y_i]\!]$.

As we let (U,h) vary over slope data for $\mathscr{X}^K_{\underline{D}^\times/F,\xi,\mathfrak{m}}$, we see by Lemma 5.2.3 that the patched modules glue to a sheaf $\mathscr{M}_\infty$ on all of $X^{\square,\psi,\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty}$. Moreover, the collection $\{Z_{U,h}\}$ is a cover of $\mathscr{Z}$ by [JN16, Theorem 2.3.2(2)], so $\mathscr{M}_\infty$ is a coherent sheaf on $\mathscr{Z}\times\operatorname{Spa}\mathbf{Z}_p[\![y_i]\!]$.

Now we have a sequence of morphisms

$$X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty}\hookrightarrow X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}}\times\times\operatorname{Spa}\mathbf{Z}_p[\![x_i]\!]\to\mathscr{Z}\times\operatorname{Spa}\mathbf{Z}_p[\![y_i]\!]$$

(where we send the product of the factors of $\mathbf{G}_m^{\mathrm{ad}}$ in the definition of the trianguline varieties to the factor of $\mathbf{G}_m^{\mathrm{ad}}$ in the definition of $\mathscr{Z}$, corresponding to the action of U_p^{-1}); $\mathscr{M}_\infty$ is a sheaf on $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty}\times\mathbf{G}_m^{\mathrm{ad}}$ whose pushforward to $\mathscr{Z}\times\operatorname{Spa}\mathbf{Z}_p[\![y_i]\!]$ is coherent. It follows that $\mathscr{M}_\infty$ is coherent over $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty}\times\mathbf{G}_m^{\mathrm{ad}}$ and $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}}\times\mathbf{G}_m^{\mathrm{ad}}\times\operatorname{Spa}\mathbf{Z}_p[\![x_i]\!]$, and in fact, over closed subspaces isomorphic to $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty}$ and $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}}$, respectively.

We summarize this discussion:

Theorem 5.2.4. *There is a space $\mathscr{X}^\infty_{\underline{D}^\times}$ (which we call the patched eigenvariety), a coherent sheaf $\mathscr{M}_\infty$ supported on all of $\mathscr{X}^\infty_{\underline{D}^\times}$, (which we call the patched module) and a morphism*

$$\mathscr{X}^\infty_{\underline{D}^\times,\leq h}\to\operatorname{Spa}X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}}[\![x_i]\!]$$

whose image is the union of irreducible components.

Since this morphism factors through the global trianguline variety, we also deduce the following corollary:

Corollary 5.2.5. *The support of $\mathscr{M}_\infty/(y_1,\dots,y_k)$ in the trianguline variety over $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\infty}/(y_1,\dots,y_k)\cong X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho}}$ is a union of irreducible components.*

5.3. **Modularity.** We are now in a position to prove Theorem 1.

Proposition 5.3.1. *Let $F/\mathbf{Q}$ be a real quadratic extension split at p, such that the image of $\overline{\rho}|_{\mathrm{Gal}_F}$ contains $\mathrm{SL}_2(\mathbf{F}_p)$. Then $\rho:\mathrm{Gal}_{\mathbf{Q}}\to\mathrm{GL}_2(L)$ is modular if and only if $\rho|_{\mathrm{Gal}_F}$ is modular.*

Proof. We have the cyclic base-change morphism $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q},\mathrm{cusp}}\to\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}}$ from §3.4, so if ρ corresponds to $x\in\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q},\mathrm{cusp}}$, then $\rho|_{\mathrm{Gal}_F}$ corresponds to the image of x in $\mathscr{X}_{\mathrm{GL}_2/F,\mathrm{mid}}$. To show the other direction, we note that if $\rho|_{\mathrm{Gal}_F}$ is associated to $x'\in\mathscr{X}_{\mathrm{GL}_2/F}$, then the corresponding eigenvalues are fixed by $\mathrm{Gal}(F/\mathbf{Q})$. Since we assumed that the image of $\overline{\rho}|_{\mathrm{Gal}_F}$ contains $\mathrm{SL}_2(\mathbf{F}_p)$, by [BH17, Theorem B.0.1] we may apply Corollary 3.4.8 to conclude that x' is in the image of $\mathscr{X}_{\mathrm{GL}_2/\mathbf{Q},\mathrm{cusp}}$. $\square$

Choose $F/\mathbf{Q}$ a real quadratic extension split at p. We may additionally choose F such that the image of $\overline{\rho}|_{\mathrm{Gal}_F}$ contains $\mathrm{SL}_2(\mathbf{F}_p)$, by requiring that

ℓ splits in F for ℓ in some finite set of primes S of $\mathbf{Q}$ such that $\{\overline{\rho}(\mathrm{Frob}_\ell)\}_{\ell\in S}$ generate $\mathrm{SL}_2(\mathbf{F}_p)$. Maintaining the notation of the previous section, we let D/F be a totally definite quaternion algebra, split at all finite places, and we let $R := \mathscr{O}_E[\![T_0/\overline{Z(K)}]\!]$. The Jacquet–Langlands correspondence gives us a morphism of eigenvarieties $\mathscr{X}_{\underline{D}^\times} \to \mathscr{X}_{\mathrm{GL}_2/F}$, so it suffices to show that $\rho|_{\mathrm{Gal}_F}$ corresponds to a point on $\mathscr{X}_{\underline{D}^\times}$.

Theorem 5.3.2. *$\rho|_{\mathrm{Gal}_F}$ corresponds to a point on $\mathscr{X}_{\underline{D}^\times}$.*

Proof. Let $\rho_0 := \rho|_{\mathrm{Gal}_F}$. We have assumed that $\rho|_{\mathrm{Gal}_{\mathbf{Q}_p}}$ is trianguline, so we may write $D_{\mathrm{rig}}(\rho|_{\mathrm{Gal}_{\mathbf{Q}_p}})$ as an extension of rank-1 (φ,Γ)-modules:

$$0 \to \Lambda_{L,\mathrm{rig}}(\delta_1) \to D_{\mathrm{rig}}(\rho|_{\mathrm{Gal}_{\mathbf{Q}_p}}) \to \Lambda_{L,\mathrm{rig}}(\delta_2) \to 0$$

for characters $\delta_1, \delta_2 : \mathbf{Q}_p^\times \rightrightarrows L^\times$. We fix a weight κ_0 according to $\delta_1|_{\mathbf{Z}_p^\times}$ and $\delta_2|_{\mathbf{Z}_p^\times}$, and we fix an unramified character $\psi_0 : \mathrm{Gal}_F \to \mathscr{O}_E[\![T_0/\overline{Z(K)}]\!]$ deforming $\chi_{\mathrm{cyc}}\kappa_{0,1}\kappa_{0,2}\det\rho$.

It is enough to show that the point $x_0 \in X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho}_0}$ corresponding to ρ_0 is in the support of $\mathscr{M}_\infty/(y_1,\dots,y_k)$. Since the parameters of $D_{\mathrm{rig}}(\rho)$ were assumed regular, x_0 is a smooth point of $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho}_0,\mathrm{loc},\leq h}$. Therefore, x_0 is contained in a unique irreducible component V of $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho}_0,\mathrm{loc},\leq h}$, and ρ_0 can be analytically deformed to characteristic 0 (as in Example 2.3.4).

Recall that for any p-adic field $K/\mathbf{Q}_p$, given a character $\delta : K^\times \to \overline{\mathbf{Q}}_p^\times$, its *weight* $(\mathrm{wt}_\sigma(\delta))_{\sigma:K\hookrightarrow\overline{\mathbf{Q}}_p}$ is the tuple such that

$$\lim_{a\to 0}\frac{|\delta(1+a)-1+\sum_\sigma \mathrm{wt}_\sigma(\delta)\sigma(a)|}{|a|} = 0$$

We say that δ is *locally algebraic of weight* $(k_\sigma)_\sigma$ if $\mathrm{wt}_\sigma(\delta) = k_\sigma \in \mathbf{Z}$ for all σ; equivalently, the restriction of δ to some open subgroup of $\mathscr{O}_K^\times$ is $x \mapsto \prod_\sigma \sigma(x)^{-k_\sigma}$. If $\underline{\delta}$ is the parameter of a trianguline (φ,Γ)-module, we say that it is *locally algebraic of strongly dominant weight* if $\delta_{i,\sigma}$ is locally algebraic of weight $(k_{i,\sigma})$ and $k_{i,\sigma} < k_{i+1,\sigma}$, for all i and σ.

Locally algebraic strongly dominant weights are very Zariski-dense in $\mathscr{W}_F^{\mathrm{rig}}$. In our setting, this means that there is some locally algebraic strongly dominant $\kappa_1 \in \mathscr{W}_F^{\mathrm{rig}}$ such that the fiber $V^{\mathrm{rig}}|_{\kappa_1}$ is 8-dimensional. By [BC09, Proposition 2.3.4], this implies that on a Zariski dense open subspace of $V^{\mathrm{rig}}|_{\kappa_1}$, the corresponding Galois representation is de Rham (for both places of F above p).

We observe that V is in the image of $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho},\mathrm{loc}}$, because ρ_0 is a global Galois representation, and the image is a union of irreducible components of $X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho}_0,\mathrm{loc},\leq h}$. Inspection of the patching construction shows that the image of $\cup_n X^{\square,\psi',\underline{\kappa}}_{\mathrm{tri},\overline{\rho}_0,Q'_n,\leq h}$ yields a Zariski dense subset of the fiber of V over κ_1.

Thus, we obtain a representation $\rho_1 : \mathrm{Gal}_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$, unramified outside $Q_n \cup \Sigma_p$ for some n and de Rham of regular Hodge–Tate weights at places above p, corresponding to a smooth point of $V|_{\kappa_1}$. But then ρ_1 is known to be modular, so V is in the support of M_∞ and we are done. □

Appendix A. Extensions of Zariski-closed subsets

The paper [Lou17] proves Riemann extension theorems for functions on normal pseudorigid spaces and normal excellent formal schemes; in this appendix we use those results to extend Zariski-closed adic subsets (in the sense of [JN19a, §2.1]) of pseudorigid spaces over missing subsets of codimension at least 2.

Proposition A.0.1. *Let X be a normal pseudorigid space and let $U \subset X$ be a Zariski-open subspace whose complement has codimension at least 2. If $Z \subset U$ is a Zariski-closed adic subset, then there is a Zariski-closed adic subset $Z' \subset X$ such that $Z' \cap U = Z$.*

Proposition A.0.2. *Let $\mathfrak{X}$ be a normal excellent formal scheme, which is nowhere discrete. If $Z \subset X := \mathfrak{X}^{\mathrm{an}}$ is a Zariski-closed adic subset, then there is a closed formal subscheme $\mathfrak{Z} \subset \mathfrak{X}$ such that $Z = \mathfrak{Z}^{\mathrm{an}}$.*

As the proofs are essentially identical, we only give a proof of the latter:

Proof of Proposition A.0.2. We may assume that $\mathfrak{X} = \mathrm{Spf}\, R$, where R is a normal excellent domain with ideal of definition $J = (f_1, \ldots, f_n)$. Then by the definitions of [JN19a, §2.1], there is a coherent sheaf $\mathcal{I} \subset \mathscr{O}_X$ of ideals such that $Z = \{x \in X \mid \mathcal{I}_x \neq \mathscr{O}_{X,x}\}$. We need to show that there is an ideal $I \subset R$ whose associated sheaf agrees with $\mathcal{I}$ on X.

We define $\mathcal{I}^+ := \mathcal{I} \cap \mathscr{O}_X^+$, and we set $I := \Gamma(X, \mathcal{I}^+)$; by [Lou17, Proposition 6.2], $R = \Gamma(X, \mathscr{O}_X^+)$, so we may view I as an ideal of R. It remains to show that for each affinoid open subspace $\mathrm{Spa}\, R' \subset X$, $R' \otimes_R I = \mathcal{I}(\mathrm{Spa}\, R')$. To see this, we observe that we have a finite cover $X = \cup_i \mathrm{Spa}\, R\left\langle \frac{J}{f_i} \right\rangle$, so it suffices to check this with $R' = \mathrm{Spa}\, R\left\langle \frac{J}{f_i} \right\rangle$.

Setting $R_i := R\left\langle \frac{J}{f_i} \right\rangle$ and $U_i = \mathrm{Spa}\, R\left\langle \frac{J}{f_i} \right\rangle$, we have an exact sequence of R-modules

$$0 \to I \to \prod_i \mathcal{I}^+(U_i) \rightrightarrows \prod_{i,j} \mathcal{I}^+(U_i \cap U_j)$$

For any fixed index i_0, we may tensor with $R_{i_0}^\circ$ and complete f_{i_0}-adically; as R is noetherian, our sequence

$$0 \to R_{i_0}^\circ \mathop{\widehat{\otimes}}_R I \to \prod_i \left(R_{i_0}^\circ \mathop{\widehat{\otimes}}_R \mathcal{I}^+(U_i) \right) \rightrightarrows \prod_{i,j} \left(R_{i_0}^\circ \mathop{\widehat{\otimes}}_R \mathcal{I}^+(U_i \cap U_j) \right)$$

remains exact. But $R_{i_0}^\circ \widehat{\otimes}_R \mathcal{I}^+(U_i)$ generates $\mathcal{I}(U_{i_0} \cap U_i)$ and $R_{i_0}^\circ \widehat{\otimes}_R \mathcal{I}^+(U_i \cap U_j)$ generates $\mathcal{I}(U_{i_0} \cap U_i \cap U_j)$ after inverting a pseudo-uniformizer u_{i_0} of R_{i_0} for all i, j, and $\{U_{i_0} \cap U_i\}_i$ is a cover of U_{i_0}, so in fact $R_{i_0} \widehat{\otimes}_R I = \mathcal{I}(U_{i_0})$, as desired. □

Corollary A.0.3. *Let R_1, R_2 be complete local noetherian $\mathscr{O}_E$-algebras, where $\mathscr{O}_E$ is the ring of integers in some p-adic field E, and suppose that R_1 and R_2 have dimension at least 2. If $Z \subset (\operatorname{Spa} R_1)^{\mathrm{an}} \times_{\mathscr{O}_E} (\operatorname{Spa} R_2)^{\mathrm{an}}$ is a Zariski-closed adic subset, then there is a is a closed formal subscheme $\mathfrak{Z} \subset \operatorname{Spf} R_1 \widehat{\otimes}_{\mathscr{O}_E} R_2$ such that $\mathfrak{Z}^{\mathrm{an}} \cap (\operatorname{Spa} R_1)^{\mathrm{an}} \times_{\mathscr{O}_E} (\operatorname{Spa} R_2)^{\mathrm{an}} = Z$ (where the intersection is taken inside $\left(\operatorname{Spa} R_1 \widehat{\otimes} R_2\right)^{\mathrm{an}}$).*

Proof. If R_1 and R_2 are power series rings over $\mathscr{O}_E$, then the result follows by combining Propositions A.0.1 and A.0.2. Indeed, we may write $R_1 = \mathscr{O}_E[\![x_1, \ldots, x_{n_1}]\!]$ and $R_2 = \mathscr{O}_E[\![x_{n_1+1}, \ldots, x_{n_1+n_2}]\!]$ with $n_1, n_2 \geq 1$; then the complement of

$$(\operatorname{Spa} R_1)^{\mathrm{an}} \times (\operatorname{Spa} R_2)^{\mathrm{an}} \subset \left(\operatorname{Spa} R_1 \widehat{\otimes} R_2\right)$$

is the disjoint union of $\{p = x_1 = \ldots = x_{n_1} = 0\}$ (which has codimension $n_1 + 1 \geq 2$) and $\{p = x_{n_1+1} = \ldots = x_{n_1+n_2} = 0\}$ (which has codimension $n_2 + 1 \geq 2$).

We will reduce the general case to the case of power series rings. We may write $R_1 = \mathscr{O}_E[\![x_1, \ldots, x_{n_1}]\!]/I_1$ and $R_2 = \mathscr{O}_E[\![x_{n_1+1}, \ldots, x_{n_1+n_2}]\!]/I_2$ with $n_1, n_2 \geq 1$; then $\left(\operatorname{Spa} \mathscr{O}_E[\![\{x_i\}_{i=1}^{n_1}]\!] \widehat{\otimes} \mathscr{O}_E[\![\{x_i\}_{i=n_1+1}^{n_2}]\!]\right)^{\mathrm{an}}$ is covered by the pseudoaffinoid spaces

$$\mathcal{V}_0 := \operatorname{Spa} E \left\langle \left\{ \frac{x_i}{p} \right\}_{i=1}^{n_1+n_2} \right\rangle$$

and

$$\mathcal{V}_{i_0} := \operatorname{Spa} \mathscr{O}_E[\![x_i]\!] \left\langle \frac{p}{x_{i_0}}, \left\{ \frac{x_i}{x_{i_0}} \right\}_{i=1}^{n_1+n_2} \right\rangle \left[\frac{1}{x_{i_0}} \right]$$

for $i_0 = 1, \ldots, n_1+n_2$, and $(\operatorname{Spa} \mathscr{O}_E[\![\{x_i\}_{i=1}^{n_1}]\!])^{\mathrm{an}} \times_{\mathscr{O}_E} \left(\operatorname{Spa} \mathscr{O}_E[\![\{x_i\}_{i=n_1+1}^{n_2}]\!]\right)^{\mathrm{an}}$ is covered by the pseudoaffinoid spaces

$$\mathcal{U}_0 := \operatorname{Spa} E \left\langle \left\{ \frac{x_i}{p} \right\}_{i=1}^{n_1+n_2} \right\rangle = \mathcal{V}_0$$

and

$$\mathcal{U}_{i_0 j_0} := \operatorname{Spa} \mathscr{O}_E[\![x_i]\!] \left\langle \left\{ \frac{x_i}{x_{i_0}} \right\}_{i=1}^{n_1+n_2}, \frac{p}{x_{j_0}}, \left\{ \frac{x_j}{x_{j_0}} \right\}_{j=n_1+1}^{n_1+n_2} \right\rangle \left[\frac{1}{x_{i_0}}, \frac{1}{x_{j_0}} \right] \subset \mathcal{V}_{i_0}$$

for $i_0 = 1, \ldots, n_1$ and $j_0 = n_1 + 1, \ldots, n_1 + n_2$, or

$$\mathcal{U}'_{i_0 j_0} := \operatorname{Spa} \mathscr{O}_E[\![x_i]\!] \left\langle \frac{p}{x_{j_0}}, \left\{ \frac{x_i}{x_{i_0}} \right\}_{i=1}^{n_1}, \left\{ \frac{x_j}{x_{j_0}} \right\}_{j=1}^{n_1+n_2} \right\rangle \left[\frac{1}{x_{i_0}}, \frac{1}{x_{j_0}} \right] \subset \mathcal{V}_{j_0}$$

for $i_0 = 1, \dots, n_1$ and $j_0 = n_1+1, \dots, n_1+n_2$. The vanishing locus of the ideal $I_1 + I_2$ on these spaces defines their intersections with $\left(\operatorname{Spa} R_1 \widehat{\otimes} R_2\right)^{\mathrm{an}}$ or $(\operatorname{Spa} R_1)^{\mathrm{an}} \times (\operatorname{Spa} R_2)^{\mathrm{an}}$, as appropriate.

We may furthermore find radical ideals

$$J_0 \subset E\left\langle \left\{\frac{x_i}{p}\right\}_{i=1}^{n_1+n_2} \right\rangle$$
$$J_{i_0j_0} \subset \mathscr{O}_E[\![x_i]\!]\left\langle \left\{\frac{x_i}{x_{i_0}}\right\}_{i=1}^{n_1+n_2}, \frac{p}{x_{j_0}}, \left\{\frac{x_j}{x_{j_0}}\right\}_{j=n_1+1}^{n_1+n_2} \right\rangle \left[\frac{1}{x_{i_0}}, \frac{1}{x_{j_0}}\right]$$
$$J'_{i_0j_0} \subset \mathscr{O}_E[\![x_i]\!]\left\langle \frac{p}{x_{j_0}}, \left\{\frac{x_i}{x_{i_0}}\right\}_{i=1}^{n_1}, \left\{\frac{x_j}{x_{j_0}}\right\}_{j=1}^{n_1+n_2} \right\rangle \left[\frac{1}{x_{i_0}}, \frac{1}{x_{j_0}}\right]$$

such that

$$Z \cap \mathcal{U}_0 = V(\widetilde{J}_0)$$
$$Z \cap \mathcal{U}_{i_0j_0} = V(\widetilde{J}_{i_0j_0})$$
$$Z \cap \mathcal{U}'_{i_0j_0} = V(\widetilde{J}'_{i_0j_0})$$

where $\widetilde{J}_0$, $\widetilde{J}_{i_0j_0}$, and $\widetilde{J}'_{i_0j_0}$ are the coherent sheaves of ideals associated to J_0, $J_{i_0j_0}$, and $J'_{i_0j_0}$, respectively.

By [JN19a, Proposition 2.2.9], $\widetilde{J}_{i_0j_0}|_{\mathcal{U}_{i_0j_0}\cap\mathcal{U}_{i_0j'_0}} = \widetilde{J}_{i_0j'_0}|_{\mathcal{U}_{i_0j_0}\cap\mathcal{U}_{i_0j'_0}}$ for all $j_0, j'_0 = n_1+1, \dots, n_1+n_2$ (and similarly for $\widetilde{J}'_{i_0j_0}|_{\mathcal{U}_{i_0j_0}\cap\mathcal{U}_{i'_0j_0}}$), because

$$V(\widetilde{J}_{i_0j_0}|_{\mathcal{U}_{i_0j_0}\cap\mathcal{U}_{i_0j'_0}}) = Z \cap \mathcal{U}_{i_0j_0} \cap \mathcal{U}_{i_0j'_0} = V(\widetilde{J}_{i_0j'_0}|_{\mathcal{U}_{i_0j_0}\cap\mathcal{U}_{i_0j'_0}})$$

Therefore, we obtain coherent sheaves of radical ideals $\mathcal{J}_{i_1}$ on $\cup_{j_0=n_1+1}^{n_1+n_2}\mathcal{U}_{i_0j_0}$. By a similar argument, we obtain a coherent sheaf $\mathcal{J}$ of radical ideals on all of

$$(\operatorname{Spa} \mathscr{O}_E[\![\{x_i\}_{i=1}^{n_1}]\!])^{\mathrm{an}} \times_{\mathscr{O}_E} \left(\operatorname{Spa} \mathscr{O}_E[\![\{x_i\}_{i=n_1+1}^{n_2}]\!]\right)^{\mathrm{an}} = \mathcal{U}_0 \cup \bigcup_{j_0=n_1+1}^{n_1+n_2} \mathcal{U}_{i_0j_0} \cup \bigcup_{i_0=1}^{n_1} \mathcal{U}'_{i_0j_0}$$

But this puts us precisely in the situation treated above, when R_1 and R_2 are power series rings. □

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