TRIANGULINE LIFTS IN POSITIVE CHARACTERISTIC

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ABSTRACT. We show that if $\overline{\rho} : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ is a continuous Galois representation, then it admits a lift $\rho : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p(\!(u)\!))$ which is trianguline. We use the families of crystalline Galois representations constructed in [BLJZ04] in characteristic 0 to construct the desired lifts.

1. Introduction

Let $\overline{\rho} : \mathrm{Gal}_K \to \mathrm{GL}_d(\overline{\mathbf{F}}_p)$ be a continuous Galois representation. Over the past few years, the trianguline variety associated to $\overline{\rho}$ has become important, and we wish to understand its global structure. Trianguline Galois representations are continuous representations $\rho : \mathrm{Gal}_K \to \mathrm{GL}_d(E)$, where E is a non-archimedean local field, whose associated (φ, Γ)-module is an extension of rank-1 objects. The trianguline variety associated to $\overline{\rho}$ is the moduli space of trianguline representations, together with the (ordered) parameters of its rank-1 subquotients, which are lifts of $\overline{\rho}$.

If we drop the condition on the residual representation and only study the moduli space of triangulated (φ, Γ)-modules, its structure is fairly straightforward. However, as is common in p-adic Hodge theory, introducing integral structures makes any analysis much more complicated. There are a number of natural questions which seem difficult to answer, such as what the irreducible components are, and whether each one contains a characteristic p point.

In this short note, we begin to answer some of these questions in the case $K = \mathbf{Q}_p$ and $d = 2$. We show that for any $\overline{\rho}$, there is a trianguline Galois representation $\rho : \mathrm{Gal}_{\mathbf{Q}_p} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p(\!(u)\!))$ which is a lift of $\overline{\rho}$. To do this, we will use the results of [BLJZ04] on Wach modules and reductions of crystalline Galois representations.

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2. The reducible case

Suppose first that $\overline{\rho} = \begin{pmatrix} \overline{\chi}_1 & * \\ & \overline{\chi}_2 \end{pmatrix}$ for some characters $\overline{\chi}_1, \overline{\chi}_2 : \mathrm{Gal}_{\mathbf{Q}_p} \to \overline{\mathbf{F}}_p^\times$. Then we may simply take the isotrivial deformation of $\overline{\rho}$ over $\overline{\mathbf{F}}_p[\![u]\!]$.

3. The irreducible case

Suppose now that $\overline{\rho}$ is irreducible. Let $I_w \subset I$ be the wild inertia subgroup and let I_t be its tame quotient. Pro-p groups acting on finite-dimensional $\mathbf{F}_p$-vector spaces always have non-trivial fixed vectors, so $\overline{\rho}|_{I_w}$ fixes at least a 1-dimensional subspace. Since I_w is normal in I, $\overline{\rho}|_I$ is reducible. Finally, if g is a lift of Frobenius and $h \in I_w$, then $\overline{\rho}(g^{-1}hg)$ still has p-power order, hence is the image of $h' \in I_w$; it follows that $\mathrm{Gal}_{\mathbf{Q}_p}$ preserves the subspace fixed by I_w, and since $\overline{\rho}$ is assumed irreducible, we must have $\overline{\rho}|_I$ trivial.

Now since I_t is abelian and has prime-to-p order, we must have that $\overline{\rho}|_I$ is semisimple and the sum of two characters, i.e., $\overline{\rho}|_I \cong \chi_1 \oplus \chi_2$. If $\chi_1 = \chi_2$, then $\overline{\rho}$ is reducible; we have already handled that case, so we assume the χ_i are distinct. Since the conjugation action of Frobenius on I_t is "raise to the pth power", we have $\{\chi_1, \chi_2\} = \{\chi_1^p, \chi_2^p\}$, so either $\chi_i^p = \chi_i$ or $\chi_i^p = \chi_{i+1} \neq \chi_i$ for $i = 1, 2$. In the first case, the χ_i are valued in $\mathbf{F}_p^\times$ and (up to unramified twist) are powers of the cyclotomic character; in the second case, the χ_i are valued in $\mathbf{F}_{p^2}^\times$ and (up to unramified twist) are powers of the fundamental characters ω_2, ω_2' of level 2.

In either case, $\chi_i^{p^2} = \chi_i$. It follows that if $\chi_1 \neq \chi_2$, then $\overline{\rho}|_{\mathrm{Gal}_{\mathbf{Q}_{p^2}}}$ is the direct sum of two characters, i.e., $\overline{\rho}|_{\mathrm{Gal}_{\mathbf{Q}_{p^2}}} \cong \left(\begin{smallmatrix} \mathrm{un}_{\alpha_1}\chi_1 & \\ & \mathrm{un}_{\alpha_2}\chi_2 \end{smallmatrix}\right)$, where un_{α_i} are the unramified characters of $\mathrm{Gal}_{\mathbf{Q}_{p^2}}$ sending Frobenius to $\alpha_i \in \overline{\mathbf{F}}_p^\times$. If $\mathrm{Frob_p}$ is a lift of the Frobenius of $\mathrm{Gal}_{\mathbf{Q}_p}$, then conjugating $\overline{\rho}|_I \cong \left(\begin{smallmatrix} \chi_1 & \\ & \chi_2 \end{smallmatrix}\right)$ by $\overline{\rho}(\mathrm{Frob_p})$ either preserves the order of the χ_i or swaps them. We may assume the second case, since otherwise $\overline{\rho}$ would again be reducible, which forces $\overline{\rho}(\mathrm{Frob_p})$ to be anti-diagonal.

In addition, $\overline{\rho}(\mathrm{Frob_p})$ squares to $\left(\begin{smallmatrix} \alpha_1 & \\ & \alpha_2 \end{smallmatrix}\right)$ and commutes with it. This, in turn, forces $\alpha_1 = \alpha_2$. Thus, up to unramified twist, we have $\overline{\rho} \cong \mathrm{Ind}_{\mathrm{Gal}_{\mathbf{Q}_{p^2}}}^{\mathrm{Gal}_{\mathbf{Q}_p}} \chi$ for some totally ramified character $\chi : \mathrm{Gal}_{\mathbf{Q}_{p^2}} \to \mathbf{F}_{p^2}^\times$.

We may write $\chi = \omega_2^a {\omega_2'}^b$, where $0 \leq a, b \leq p-1$. Relabelling if necessary, we may assume that $b \leq a$, so that $\chi = \chi_{\mathrm{cyc}}^b \omega_2^{a-b}$, and again $0 \leq a-b \leq p-1$. Since χ_{cyc}^b extends to a character of $\mathrm{Gal}_{\mathbf{Q}_p}$ (after possibly extending the coefficients to contain a square root of α), we have $\mathrm{Ind}_{\mathrm{Gal}_{\mathbf{Q}_{p^2}}}^{\mathrm{Gal}_{\mathbf{Q}_p}} \chi = \chi_{\mathrm{cyc}}^b \otimes \mathrm{Ind}_{\mathrm{Gal}_{\mathbf{Q}_{p^2}}}^{\mathrm{Gal}_{\mathbf{Q}_p}} \omega_2^{a-b}$ and it suffices to produce a trianguline lift of $\mathrm{Ind}_{\mathrm{Gal}_{\mathbf{Q}_{p^2}}}^{\mathrm{Gal}_{\mathbf{Q}_p}} \omega_2^{a-b}$.

Proposition 3.0.1. *For any $0 \leq k \leq p-1$, there is a trianguline Galois representation $\rho : \mathrm{Gal}_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p(\!(u)\!))$ such that $\overline{\rho} = \mathrm{Ind}_{\mathrm{Gal}_{\mathbf{Q}_{p^2}}}^{\mathrm{Gal}_{\mathbf{Q}_p}} \omega_2^k$.*

Proof. In [BLJZ04, §3], the authors constructed a family of Wach modules $N(u)$ over $R := \mathbf{Z}_{p^2}[\![u]\!]$, with

$$\varphi = \begin{pmatrix} 0 & -1 \\ q^k & uz \end{pmatrix}$$

for some explicit $z \in \mathbf{Z}_p[\![\pi]\!]$. They show that it induces a family of étale (φ, Γ)-modules over $\mathbf{Z}_{p^2}[\![u]\!]$; it arises from a family of Galois representations, by work of Dee, and modulo (p, u) it is $\mathrm{Ind}_{\mathrm{Gal}_{\mathbf{Q}_{p^2}}}^{\mathrm{Gal}_{\mathbf{Q}_p}} \omega_2^k$. Reducing modulo π, we obtain a family of lattices in crystalline representations with Hodge–Tate weights $\{0, -k\}$.

If we consider φ acting on D_{cris} of the corresponding family of Galois representations, that is, the rigid analytic generic fiber of $N(u)$ modulo π, it has characteristic polynomial $P(\lambda) := \lambda^2 - u\overline{z}\lambda + q^k$. We claim that $\overline{z} = 1$. Indeed, set $q := \varphi(\pi)/\pi$, $q_n := \varphi^{n-1}(q)$,

$$\lambda_+ := \prod_{n\geq 0} \frac{\varphi^{2n+1}(q)}{p}$$

and

$$\lambda_- := \prod_{n\geq 0} \frac{\varphi^{2n}(q)}{p}$$

Then if $\lambda_-/\lambda_+ =: \sum_{n\geq 0} z_n\pi^n$, z is by definition $z_0 + z_1\pi + \cdots + z_{a-2}z^{a-2}$. By calculation, q_n/p is a polynomial in π of degree $p^{n-1}(p-1)$ with leading coefficient $1/p$, constant term 1, and all other terms with integral coefficients. It follows that p/q_n is a power series in $\mathbf{Q}_p[\![\pi]\!]$ with constant term 1, and hence that $z_0 = 1$.

Thus, over the locus of $\operatorname{Spf}\mathbf{Z}_{p^2}[\![u]\!]$ with $v_p(u) \leq \frac{k}{2}$, the eigenvalues of Frobenius have valuations $v_p(u)$ and $k - v_p(u)$, respectively. The corresponding (φ,Γ)-module is trianguline at all points of $\operatorname{Spf}\mathbf{Z}_{p^2}[\![u]\!]$ (since the Galois representation is crystalline), with triangulations given by a choice of orderings of eigenvalues of Frobenius.

We pull our family of Galois representations back to the cover X of $\operatorname{Spf}\mathbf{Z}_{p^2}[\![u]\!]^{\text{an}}$ defined by P; P splits over this cover with roots λ_1, λ_2, and we may assume that $\lambda_1 \equiv u \neq 0 \pmod p$. Now we choose the triangulation over X given by

$$\delta_1|_{\mathbf{Z}_p^\times} = \text{id} \qquad , \qquad \delta_1(p) = \lambda_1$$

and

$$\delta_2|_{\mathbf{Z}_p^\times} = \chi_{\text{cyc}}^{-k} \qquad , \qquad \delta_2(p) = p^{-k}\lambda_2$$

By [Bel23, Theorem 1.4], it follows that the specialization of $N(u)$ at $p = 0$ induces a trianguline (φ,Γ)-module. This is the desired lift. □

References

[Bel23] Rebecca Bellovin, *Cohomology of (φ,Γ)-modules over pseudorigid spaces*, International Mathematics Research Notices (2023), rnad093.

[BLJZ04] Laurent Berger, Hanfeng Li, and Hui June Zhu, *Construction of some families of 2-dimensional crystalline representations*, Mathematische Annalen **329** (2004), 365–377.