M3/4/5P12 PROGRESS TEST #1

PLEASE WRITE YOUR NAME AND CID NUMBER ON EVERY SCRIPT THAT YOU HAND IN. FAILURE TO DO THIS MAY RESULT IN YOU NOT RECEIVING MARKS FOR QUESTIONS THAT YOU ANSWER.

Note: all representations are assumed to be on finite dimensional complex vector spaces.

Question 1. Let G be a finite group.

(1) Let (V, ρ_V) be a representation of G. Define a subrepresentation of V.
(2) If (V, ρ_V) is a representation of G, recall that $V^G := \{v \in V | \rho_V(g)(v) = v \text{ for all } g \in G\}$. Show that V^G is a subrepresentation of V.
(3) Show that there is a G-linear map $\pi : V \to V^G$ such that $\pi|_{V^G}$ is the identity map.
(4) Let $G = C_4$, and consider the 3-dimensional representation (V, ρ_V) defined by

$$\rho_V(g) = \begin{pmatrix} i & 0 & 0 \\ i-1 & 1 & 0 \\ i-1 & 0 & 1 \end{pmatrix}$$

You do not need to prove that this is a representation. What is V^G? Using part (3), write down the decomposition of (V, ρ_V) into irreducible subrepresentations.

Question 2. Consider the symmetric group on four elements S_4. It is generated by the transpositions $\{(a\ b)\}$, where $a, b \in \{1, 2, 3, 4\}$. Let $V = \mathbf{C}^4$. There is a four-dimensional permutation representation (V, ρ), given by $\rho(\sigma)(v_i) := v_{\sigma(i)}$, where v_1, v_2, v_3, v_4 are the standard basis of $\mathbf{C}^4$. In terms of matrices,

$$\rho(1\ 2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho(1\ 3) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho(1\ 4) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho(2\ 3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \rho(2\ 4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \rho(3\ 4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

You do not need to check that this defines a representation.

(1) Show that there are two 1-dimensional representations of S_4 and write them down. It is enough to give their values on transpositions.
(2) Find a 1-dimensional subrepresentation $W \subset V$. Are there any others?
(3) Deduce that V is the direct sum of a 1-dimensional representation and an irreducible 3-dimensional representation W'. You do not need to find W'.
(4) Show that S_4 has two irreducible 1-dimensional representations, one irreducible 2-dimensional representation, and two irreducible 3-dimensional representations. You may assume any results you need from the course.