M3/4/5P12 Solutions #5

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1. If $f : G \to H$ is a homomorphism of finite groups, we define a map $\mathbf{C}[G] \to \mathbf{C}[H]$ via $[g] \mapsto [f(g)]$. To check it is an algebra homomorphism, we note that $[e] \mapsto [f(e)] = [e]$ and $[g_1 g_2] \mapsto [f(g_1 g_2)] = [f(g_1)f(g_2)] = [f(g_1)][f(g_2)]$. Since multiplication is bilinear and $\{[g]\}_{g \in G}$ is a basis for $\mathbf{C}[G]$, we are done.

2. Recall that we can produce such an isomorphism by writing down all of the irreducible representations (V_i, ρ_{V_i}) of D_8, writing down the maps $\rho_i : \mathbf{C}[D_8] \to \mathrm{Hom}(V_i, V_i)$, and considering the direct sum $\rho := \oplus_i \rho_i : \mathbf{C}[G] \to \oplus_i \mathrm{Hom}(V_i, V_i)$. We showed it is injective, and both sides have the same dimension over $\mathbf{C}$, so it is an isomorphism.

   Recall that D_8 has four irreducible 1-dimensional representations and one irreducible 2-dimensional representation. The 1-dimensional representations are given by $s, t \rightrightarrows \{\pm 1\}$, and the 2-dimensional representation is given by $s \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $t \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ (with respect to some choice of basis). Thus, we define a map

$$\begin{aligned} \mathbf{C}[D_8] &\to \mathrm{Mat}_1(\mathbf{C}) \oplus \mathrm{Mat}_1(\mathbf{C}) \oplus \mathrm{Mat}_1(\mathbf{C}) \oplus \mathrm{Mat}_1(\mathbf{C}) \oplus \mathrm{Mat}_2(\mathbf{C}) \\ s &\mapsto (1, 1, -1, -1, \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)) \\ t &\mapsto (1, -1, 1, -1, \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)) \end{aligned}$$

3. Since the quaternion group Q_8 also has four 1-dimensional representations and one irreducible 2-dimensional representation (by the previous problem sheet), $\mathbf{C}[Q_8] \cong \mathrm{Mat}_1(\mathbf{C}) \oplus \mathrm{Mat}_1(\mathbf{C}) \oplus \mathrm{Mat}_1(\mathbf{C}) \oplus \mathrm{Mat}_1(\mathbf{C}) \oplus \mathrm{Mat}_2(\mathbf{C}) \cong \mathbf{C}[D_8]$. But $Q_8 \not\cong D_8$ (again by the previous problem sheet).

4. Suppose that V is 1-dimensional as a complex vector space and V has the structure of a $\mathrm{Mat}_2(\mathbf{C})$-module. Then the basis elements $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ act on V by multiplication by scalars (respectively, $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22} \in \mathbf{C}$).

   Observe that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The axioms of an A-module require $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ to act as the 0 map on V, and also as multiplication by $\lambda_{11}\lambda_{22}$. Thus, at least one of λ_{11}, λ_{22} is 0. Moreover, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so the identity acts as multiplication by $\lambda_{11} + \lambda_{22}$, and so one of λ_{11}, λ_{22} is non-zero and equal to 1.

   Moreover, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This implies

$$\lambda_{11} = \lambda_{12}\lambda_{21} = \lambda_{21}\lambda_{12} = \lambda_{22}$$

   which contradicts the previous paragraph.

5. Consider the submodule $A \cdot x \subset A = \mathbf{C}[x]/x^3$. Since $(a_0 + a_1 x + a_2 x^2) \cdot x = a_0 x + a_1 x^2$, as a vector space $A \cdot x$ is the 2-dimensional vector subspace of A generated by x and x^2. To prove that A is not semi-simple, it suffices to prove that $A \cdot x$ does not have a complementary A-submodule inside A.

   Suppose otherwise, and let $M \subset A$ be a complement to $A \cdot x$. Since A is 3-dimensional as a vector space and $A \cdot x$ is 2-dimensional, M must be 1-dimensional. Thus, it is generated (as a vector space) by a basis element b, which we may write $b = b_0 + b_1 x + b_2 x^2 \notin A \cdot x$. If M is an A-module, then for every $a \in A$, $a \cdot b = \mu_a b \in \langle b \rangle$ for some scalar $\mu_a \in \mathbf{C}$. If we write $a = a_0 x + a_1 x^2$, then

$$a \cdot b = (a_0 x + a_1 x^2) \cdot (b_0 + b_1 x + b_2 x^2) = a_0 b_1 + (a_0 b_1 + a_1 b_0)x + (a_0 b_2 + a_1 b_1 + a_2 b_2)x^2$$

   If $a_0 \neq 0$, then we must have $\mu_a = a_0$, so $a_0 b_1 = a_0 b_1 + a_1 b_0$ and $a_0 b_2 = a_0 b_2 + a_1 b_1 + a_2 b_2$. But then if $a_1 \neq 0$, $b_0 = 0$ which contradicts our assumption that $b \notin A \cdot x$.

6. An element of $\mathrm{Hom}(M, \mathbf{C})$ is a linear map $f : M \to \mathbf{C}$. We make $\mathrm{Hom}(M, \mathbf{C})$ into an A^{op}-module by setting $(a \cdot f)(m) = f(a \cdot m)$. We need to check that this satisfies the axioms of a module. It is clear that $(a + b) \cdot f = a \cdot f + b \cdot f$, $1_A \cdot f = f$, and $a \cdot (f_1 + f_2) = a \cdot f_1 + a \cdot f_2$. Finally,

$$((ab) \cdot f)(m) = f((ab) \cdot m) = f(a \cdot (b \cdot m)) = (a \cdot f)(b \cdot m) = (b \cdot (a \cdot f))(m)$$

   Since $m^{op}(a, b) = m(ba)$, the result follows. Thus, we get a map $A^{op} \to \mathrm{Hom}(M^*, M^*)$.

   If $A = \mathbf{C}[G]$, we defined the dual representation by defining $(\rho_{M^*}(g)(f))(m) := f(\rho_M(g^{-1})(m))$. But this agrees with the composition

$$\mathbf{C}[G] \xrightarrow{\sim} \mathbf{C}[G]^{op} \to \mathrm{Hom}(M^*, M^*)$$

   where $\mathbf{C}[G] \xrightarrow{\sim} \mathbf{C}[G]^{op}$ is the map given by $[g] \mapsto [g^{-1}]$.

7. For one thing, if $A = \mathbf{C}$, this definition is not compatible with the definition of the tensor product of vector spaces. Indeed, if $a \in \mathbf{C}$, $a(m \otimes n) = (am) \otimes n = m \otimes (an)$, which is in general not equal to $(am) \otimes (an)$.