M3/4/5P12 Solutions #2

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1. (a) We first prove that the given map $G \times \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ is actually a group action. That is, we need to show that $(gh) \cdot f = g \cdot (h \cdot f)$. But

   $$\rho_W(gh) \circ f \circ \rho_V((gh)^{-1}) = \rho_W(g) \circ (\rho_W(h) \circ f \circ \rho_V(h^{-1})) \circ \rho_V(g^{-1}) = \rho_W(g) \circ (h \cdot f) \circ \rho_V(g^{-1}) = g \cdot (h \cdot f)$$

   Applying this statement with $h = g^{-1}$ implies that for every $g \in G$, the map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(V, W)$ given by $f \mapsto g \cdot f$ is invertible, so we get a representation.

   (b) Recall that f is G-linear if and only if $\rho_W(g) \circ f = f \circ \rho_V(g)$ for all $g \in G$. This holds if and only if $\rho_W(g) \circ f \circ \rho_V(g^{-1}) = f \circ \rho_V(g) \circ \rho_V(g^{-1}) = f$ for all $g \in G$, which is the same as the condition that $g \cdot f = f$ for all $g \in G$.

2. This representation is reducible, because it has a 1-dimensional subrepresentation $\langle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right) \rangle$. But this is the only 1-dimensional subrepresentation, so the representation is indecomposable.

3. Let (V, ρ) denote the given 4-dimensional representation; if v_1, v_2, v_3, v_4 are the basis corresponding to the vertices of the square, then $\rho(s)(v_i) = v_{i+1}$ (with indices taken modulo 4) and $\rho(t)(v_i) = v_{5-i}$. This representation cannot be irreducible, because we showed during lecture that the only irreducible representations of D_8 have dimension 1 or 2. We first find all the 1-dimensional subrepresentations of the given representation.

   Suppose $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$ generates a subrepresentation, so that $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$ is a simultaneous eigenvector of $\rho(s)$ and $\rho(t)$. Recall that on the first problem sheet, we showed that there are four 1-dimensional representations of D_8, and s and t act by multiplication by ± 1. Thus,

   $$\rho(s)(a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4) = a_4v_1 + a_1v_2 + a_2v_3 + a_3v_4$$

   so we must have $a_i = \pm 1 \cdot a_{i+1}$. Similarly,

   $$\rho(t)(a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4) = a_4v_1 + a_3v_2 + a_2v_3 + a_1v_4$$

   so we must have $a_i = \pm 1 \cdot a_{5-i}$.

   This implies that $a_1 = a_3$, $a_2 = a_4$, and $a_1 = \pm a_2$. We conclude that the only 1-dimensional subrepresentations of V are generated by $v_1 + v_2 + v_3 + v_4$ and $v_1 - v_2 + v_3 - v_4$.

   It follows that V is the direct sum of an irreducible 2-dimensional representation, the trivial representation, and the 1-dimensional representation $s, t \mapsto -1$.

4. Any 1-dimensional subrepresentation has to be generated by an eigenvector for both $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$. Since the eigenvalues of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ are distinct, any eigenvector is either a multiple of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$ or a multiple of $\left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix} \right)$. But neither of these is an eigenvector for $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$, so $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right)$ have no common eigenvectors and the representation is irreducible.

   This is the only irreducible 2-dimensional of D_8, up to isomorphism. To see this, either go through Exercise 8 (and note that for $n = 4$, $(n-2)/2 = 1$, or else recall from lectures that $|D_8| = d_1^1 + d_2^2 + \ldots + d_r^2$, where the d_i are the dimensions of the irreducible representations of D_8, and recall from the first problem sheet that there are exactly 4 1-dimensional representations of D_8.

5. (a) Since $\chi : G \to \mathrm{GL}_1(\mathbf{C})$ is a homomorphism and $\mathrm{GL}_1(\mathbf{C})$ is abelian, $\chi(hgh^{-1}) = \chi(h)\chi(g)\chi(h)^{-1} = \chi(g)$.
   (b) First observe that $(21a)(12)(21a)^{-1} = (1a)$. Next observe that $(1ab)(1a)(1ab)^{-1} = (ab)$. Thus, for any $a, b \in \{1, \dots, n\}$, (ab) and (12) are conjugate.
   (c) Let $\chi : S_n \to \mathrm{GL}_1(\mathbf{C})$ be a representation. Then the first two parts of this question imply that $\chi(ab) = \chi(12)$ for all $a, b \in \{1, \dots, n\}$. Since $\chi(12)^2 = 1$, we must have $\chi(12) = \pm 1$.

6. Recall that $\mathrm{Hom}(V^{\oplus r}, V^{\oplus r}) \cong \oplus_{i,j} \mathrm{Hom}(V, V)$, as representations of G. Thus, $\mathrm{Hom}(V^{\oplus r}, V^{\oplus r})^G \cong \oplus_{i,j} \mathrm{Hom}(V, V)^G$, and since $\mathrm{Hom}(V, V)^G$ is a 1-dimensional complex vector space (by Schur's lemma), $\mathrm{Hom}(V^{\oplus r}, V^{\oplus r})^G$ is a vector space of dimension r^2.

   However, $\mathrm{Hom}(V^{\oplus r}, V^{\oplus r})$ and $\mathrm{Hom}(V^{\oplus r}, V^{\oplus r})^G$ are also (non-commutative) rings, where multiplication is the composition of two endomorphisms. I claim it is isomorphic to the ring of $r \times r$ matrices. To see this, let $\delta_{ij} \in \mathrm{Mat}_r(\mathbf{C})$ denote the matrix with a 1 in the (i, j) entry and 0 everywhere else. Define a map $\mathrm{Mat}_r(\mathbf{C}) \to \mathrm{Hom}(V^{\oplus r}, V^{\oplus r})^G$ by sending δ_{ij} to the map $V^{\oplus r} \to V^{\oplus r}$ which sends the jth summand to the ith summand via the identity. Extending this by linearity defines a ring homomorphism, and it is evidently injective. Since the source and target are finite-dimensional vector spaces with the same dimension, it is an isomorphism.

7. (a) Let $G = C_n = \langle g : g^n = e \rangle$. Then a 1-dimensional matrix representation $\rho : C_n \to \mathrm{GL}_1(\mathbf{C})$ is determined by $\rho(g)$. Since $\rho(g)^n = \rho(g^n) = \rho(e) = 1$, $\rho(g)$ must be of the form ζ^i, $\zeta = e^{2\pi i/n}$. Any choice $0 \le i \le n-1$ yields a representation. Moreover, they are pairwise non-isomorphic, because if two of them were isomorphic, their matrix representations would be conjugate. But $\mathrm{GL}_1(\mathbf{C})$ is abelian, so $P\rho(g)P^{-1} = \rho(g)$ for all $P \in \mathrm{GL}_1(\mathbf{C})$.

   It suffices to show that each ζ^i is an eigenvalue of $\rho_{\mathrm{reg}}(g)$. But to see this, we simply calculate $\det(\lambda \mathbf{1} - \rho_{\mathrm{reg}}(g)) = \lambda^n - 1$.
   (b) Every finite abelian group is the product of finite cyclic groups, i.e., $G \cong \prod_i C_{n_i}$. A 1-dimensional representation $\rho : G \to \mathrm{GL}_1(\mathbf{C})$ is determined by specifying $\rho(g_i)$, where g_i is a generator of C_{n_i}. There are n_i choices for $\rho(g_i)$, so there are $\prod_i n_i$ choices for ρ. Again because $\mathrm{GL}_1(\mathbf{C})$ is abelian, they are pairwise non-isomorphic.

8. (a) The eigenvalues of $\rho_V(t)$ are two numbers of the form ζ^i, where $\zeta = e^{2\pi i/n}$. If $\rho_V(t)$ were not diagonalizable, we could put it in Jordan normal form. That is, we could find a basis of V such that $\rho_V(t)$ had matrix $\begin{pmatrix} \zeta^i & 1 \\ 0 & \zeta^i \end{pmatrix}$. But $\begin{pmatrix} \zeta^i & 1 \\ 0 & \zeta^i \end{pmatrix}^n = \begin{pmatrix} 1 & n\zeta^{n-1} \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so this is impossible. Thus, we can find a basis for V so that $\rho_V(t)$ has matrix $\begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^j \end{pmatrix}$ with $0 \le j \le i \le n-1$.
   (b) If $i = j$, then $\rho_V(t)$ is diagonal with respect to any basis of V and we may choose one diagonalizing $\rho_V(s)$.

   Otherwise, suppose $\rho_V(s)$ has matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If $a = 0$, then
   $$\rho_V(s)^2 = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} bc & bd \\ cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
   implies $d = 0$ and $bc = 1$. Similarly, assuming $d = 0$ implies that $a = 0$ and $bc = 1$.

   Now consider the equation $\rho_V(s)\rho_V(t) = \rho_V(t^{-1})\rho_V(s)$. We see
   $$a\zeta^i = a\zeta^{n-i} \qquad b\zeta^i = b\zeta^{n-j} \qquad c\zeta^j = c\zeta^{n-i} \qquad d\zeta^j = d\zeta^{n-j}$$
   Thus, if $ad \neq 0$, then $\zeta^i, \zeta^j \in \{\pm 1\}$. If $i = j$, we are in the first case we considered, so we may assume n is even and $\zeta^i = -1$, $\zeta^j = 1$. But that implies that $b = c = 0$, and $a, d \in \{\pm 1\}$, so the matrix for $\rho_V(s)$ is diagonal.

   So we may assume $a = d = 0$ and $bc = 1$. But then we see from our calculation above that $i + j = n$. Now let $P = \begin{pmatrix} b^{-1} & 0 \\ 0 & 1 \end{pmatrix}$. Then $P\rho_V(t)P^{-1} = \rho_V(t)$ and $P\rho_V(s)P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, so we may assume $b = c = 1$.
   (c) The representation is reducible in the cases above when $ad \neq 0$, and when $i = j$. When $i \neq j$ and $ad \neq 0$, the eigenspaces for $\rho_V(t)$ are distinct and generated by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Since neither of these is an eigenvector for $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the representation is irreducible.

(d) If n is even, taking $n/2 + 1 \leq i \leq n - 1$ in the above construction gives us $(n - 2)/2$ 2-dimensional irreducible matrix representations of D_{2n}. A calculation shows that no two are equivalent, so they yield non-isomorphic representations.

(e) If n is odd, taking $(n + 1)/2 \leq i \leq n - 1$ in the above construction gives us $(n - 1)/2$ 2-dimensional irreducible matrix representations of D_{2n}. A calculation shows that no two are equivalent, so they yield non-isomorphic representations.