M3/4/5P12 Solutions #1

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1. (a) Yes. Each element of G can be written uniquely in the form $s^k t^\ell$ for $0 \le k \le 3$ and $0 \le \ell \le 1$, and the image of this element under the representation we have defined is $\begin{pmatrix} (-1)^\ell & 0 \\ 0 & i^k \end{pmatrix}$. This is the identity matrix if and only if ℓ is even and $4|k$. But that implies $s^k t^\ell = e$.

   (b) No. This representation sends $s^2 t$ to
   $$\begin{pmatrix} i & 0 \\ 1 & 1 \end{pmatrix}^2 \begin{pmatrix} -1 & 0 \\ i+1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ i+1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   (c) The characteristic polynomial of Q is $\det(\lambda I - Q) = (\lambda - i)(\lambda - 1)$; it is a degree-2 polynomial with distinct roots 1 and i, so Q can be diagonalized to S. Similarly, the characteristic polynomial of R is $(\lambda-1)(\lambda+1)$, so it can be diagonalized to T.

   (d) The eigenspaces of Q are generated by $\begin{pmatrix} 2 \\ -(i+1) \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and these vectors also generate the eigenspaces of R.

   (e) No, one representation is faithful and the other isn't.

2. (a) $\rho_{\mathrm{reg}}(g) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$

   (b) The characteristic polynomial of $\rho_{\mathrm{reg}}(g)$ is $\det(\lambda I - \rho_{\mathrm{reg}}(g)) = \lambda^n - 1$, so the eigenvalues are the nth roots of 1.

   (c) Let $\zeta \in \mathbf{C}$ be an nth root of 1, so that $\zeta^n = 1$. Suppose that $a_0 b_e + a_1 b_g + \cdots + a_{n-1} b_{g^{n-1}}$ is an eigenvector for $\rho_{\mathrm{reg}}(g)$ with eigenvalue ζ. Since $\rho_{\mathrm{reg}}(g)(a_0 b_e + a_1 b_g + \cdots + a_{n-1} b_{g^{n-1}}) = a_{n-1} b_e + a_0 b_g + \cdots + a_{n-2} b_{g^{n-1}}$, $\rho_{\mathrm{reg}}(g)(a_0 b_e + a_1 b_g + \cdots + a_{n-1} b_{g^{n-1}}) = \zeta(a_0 b_e + a_1 b_g + \cdots + a_{n-1} b_{g^{n-1}})$ implies $\zeta a_i = a_{i-1}$ (with a_{-1} taken to mean a_{n-1}). Thus, $\{b_e + \zeta^{n-1} b_g + \cdots + \zeta b_{g^{n-1}}\}_\zeta$ (where the set runs over each ζ an nth root of 1) is a basis of eigenvectors.

3. Yes. Commuting diagonalizable matrices can be simultaneously diagonalized.

4. (a) (123) and (23), respectively.

   (b) The matrix for "counterclockwise rotation" is $\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$.

   (c) The eigenvalues of $\begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$ are given by the roots of $\lambda^2 + \lambda + 1$, so they are $\frac{-1 \pm i\sqrt{3}}{2}$; we write $\omega = \frac{-1+i\sqrt{3}}{2}$ and $\omega^{-1} = \omega^2 = \frac{-1-i\sqrt{3}}{2}$. The corresponding eigenspaces are generated by $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $\begin{pmatrix} -i \\ 1 \end{pmatrix}$.

   The map induced by "reflection across the x-axis" sends $\begin{pmatrix} 1 \\ -i \end{pmatrix}$ to $\begin{pmatrix} 1 \\ i \end{pmatrix} = i\begin{pmatrix} -i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ to $\begin{pmatrix} -i \\ -1 \end{pmatrix} = -i\begin{pmatrix} 1 \\ -i \end{pmatrix}$. Thus, the matrix for "reflection across the x-axis" with respect to the basis $\left\{\begin{pmatrix} 1 \\ -i \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix}\right\}$ is $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$.

5. Suppose $\rho : D_8 \to \mathrm{GL}_1(\mathbf{C})$ is a group homomorphism. Then $\rho(t)^2 = \rho(t^2) = \rho(e) = 1$, so $\rho(t) = \pm 1$. Also, $\rho(s)^{-1} = \rho(s^{-1}) = \rho(tst) = \rho(t)\rho(s)\rho(t) = \rho(s)$ (where we use the fact that $\mathrm{GL}_1(\mathbf{C})$ is a commutative group to say that $\rho(t)\rho(s)\rho(t) = \rho(t)^2\rho(s) = \rho(s)$). Thus, $\rho(s)^{-1} = \rho(s)$ so $\rho(s)^2 = 1$. Thus, there are four 1-dimensional representations of D_8.

6. (a) S_3 is generated by (123) and (23).

$$\rho(123) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \rho(23) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(b) Let (b_1, b_2, b_3) denote the standard basis for the permutation representation, and suppose that $a_1b_1 + a_2b_2 + a_3b_3$ generates a 1-dimensional subrepresentation of V; then for each $g \in S_3$ $\rho(g)(a_1b_1 + a_2b_2 + a_3b_3) = c_g(a_1b_1 + a_2b_2 + a_3b_3)$ for $c_g \in \mathbf{C}$ constants depending on g.
It is enough to consider this equation for $g = (123)$ and $g = (23)$, since those generate S_3. Then

$$\rho(123) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_3 \\ a_1 \\ a_2 \end{pmatrix} = c_{(123)} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

and

$$\rho(23) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_3 \\ a_2 \end{pmatrix} = c_{(23)} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

The second equation implies $c_{(23)} = 1$, so $a_2 = a_3$. But then the first equation implies that $c_{(123)} = 1$ and $a_1 = a_2 = a_3$.
Thus, $b_1 + b_2 + b_3$ generates a subrepresentation isomorphic to the trivial representation, and this is the only 1-dimensional subrepresentation of V.

(c) The orthogonal complement to $\langle b_1 + b_2 + b_3 \rangle$ is $W' := \{a_1b_1 + a_2b_2 + a_3b_3 : a_i \in \mathbf{C}, \sum_i a_i = 0\}$. Since this condition is preserved by permuting b_1, b_2, and b_3, W' is stabilized by S_3.

(d) The eigenvalues of $\rho(123)$ are cube roots $\{1, \omega, \omega^2\}$ of 1, and they have eigenvectors $\left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 \\ \omega^2 \\ \omega \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 1 \\ \omega \\ \omega^2 \end{smallmatrix}\right)$, respectively.
Since

$$\rho(23) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \rho(23) \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} = \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} \text{ and } \qquad \rho(23) \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$$

the matrix for $\rho(23)$ with respect to this basis of eigenvectors is still $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix}\right)$.

7. (a) We need to prove that for all $w \in W$ and $g \in G$, $f^{-1}(\rho_W(g)w) = \rho_V(g)(f^{-1}(w))$. However, $f(\rho_V(g)(f^{-1}(w)) = \rho_W(g)(f(f^{-1}(w))) = \rho_W(g)(w)$ for all $w \in W$ and $g \in G$, since f is assumed G-linear. If we apply f^{-1} to both sides, we get the desired statement.

(b) Let (V_1, ρ_1), (V_2, ρ_2), and (V_3, ρ_3) be representations of G, and suppose $f_{12} : V_1 \to V_2$ and $f_{23} : V_2 \to V_3$ are G-linear maps of vector spaces. Then for every $v \in V_1$ and every $g \in G$,

$$(f_{23} \circ f_{12})(\rho_1(g)v) = f_{23}(f_{12}(\rho_1(g)v)) = f_{23}(\rho_2(g)f_{12}(v)) = \rho_3(g)(f_{23}(f_{12}(v))) = \rho_3(g)(f_{23} \circ f_{12})(v)$$

as desired.

(c) If (V, ρ_V) is a representation, it is isomorphic to itself (since the identity map is a G-linear isomorphism). If (V, ρ_V) is isomorphic to (W, ρ_W) and $f : V \to W$ is a G-linear isomorphism, then (W, ρ_W) is isomorphic to (V, ρ_V) and $f^{-1} : W \to V$ is a G-linear isomorphism. Transitivity follows from the previous result that the composition of G-linear homomorphisms is G-linear.

8. The composition of two group homomorphisms is a group homomorphism, so $\rho \circ f : G \to \mathrm{GL}(V)$ is also a homomorphism.

9. (a) Recall that the cosets of H in G are sets of the form gH for $g \in G$. Then G acts on the set of cosets via $g' \cdot gH := (g'g)H$. Since the set of cosets is finite, we can construct a representation as in Example 3.2.5 of the notes.

(b) The cosets of A_n in S_n are A_n and $(12)A_n$; if ρ is the associated 2-dimensional representation of S_n, then for any transposition $(ab) \in S_n$, the matrix for $\rho(ab)$ with respect to the natural basis is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ (since $(ab)(12) \in A_n$). To diagonalize ρ, it is enough to diagonalize this matrix, which can be done with $P = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$.

(c) It is the regular representation of G/H, restricted to G.