M3/4/5P12: Problems about induced representations

1. We define the 1-dimensional representation (V, ρ) of C_n by setting $\rho(s) := \zeta$ for some $\zeta \in \mathbf{C}$ with $\zeta^n = 1$. Then $\operatorname{Res}^G_H \operatorname{Ind}^G_H V$ is a 2-dimensional representation, and it is isomorphic to $V_e \oplus V_t$, where $V_e \cong V$ as representations of C_n and s acts on V_t as $\rho(tst) = \rho(s)^{-1}$. Thus, $\operatorname{Ind}^G_H V$ is irreducible as a representation of $G = D_{2n}$ if and only if $\rho_V(s) \neq \rho_V(s)^{-1}$, or in other words, if $\rho_V(s) \neq \pm 1$. In particular, if n is odd, $\operatorname{Ind}^G_H V$ is irreducible unless (V, ρ_V) is the trivial representation.

2. We compute the characters of both representations. The character of $W \otimes \operatorname{Ind}^G_H V$ is

$$\chi_{W \otimes \operatorname{Ind}^G_H V}(g) = \chi_W(g) \cdot \chi_{\operatorname{Ind}^G_H V}(g) = \chi_W(g) \cdot \sum_{i : g_i g g_i^{-1} \in H} \chi_V(g_i g g_i^{-1})$$

   whereas the character of $\operatorname{Ind}^G_H(\operatorname{Res}^G_H W \otimes V)$ is

$$\chi_{\operatorname{Ind}^G_H(\operatorname{Res}^G_H W \otimes V)}(g) = \sum_{i : g_i g g_i^{-i} \in H} \chi_{\operatorname{Res}^G_H W \otimes V}(g_i g g_i^{-1}) = \sum_{i : g_i g g_i^{-i} \in H} \chi_W(g) \chi_V(g_i g g_i^{-1})$$

   Since the characters agree, the representations are isomorphic.

3. There are three irreducible representations of S_3, namely the trivial representation, the sign representation, and an irreducible 2-dimensional representation.

   In order to compute irreducible decompositions of induced representations, we will use Frobenius reciprocity. Frobenius reciprocity tells us that if (W, ρ_W) is a representation of G, then $\langle \chi_{\operatorname{Ind}^G_H V}, \chi_W \rangle = \langle \chi_V, \chi_{\operatorname{Res}^G_H W} \rangle$. We therefore write down the table of restrictions of characters of S_4 to S_3:

	$\{e\}$	$(1\ 2)$	$(1\ 2\ 3)$
size of conjugacy class	1	3	2
$\chi_{\operatorname{Res}^G_H V_{\text{triv}}}$	1	1	1
$\chi_{\operatorname{Res}^G_H V_{\text{sign}}}$	1	-1	1
$\chi_{\operatorname{Res}^G_H W}$	3	1	0
$\chi_{\operatorname{Res}^G_H W'}$	3	-1	0
$\chi_{\operatorname{Res}^G_H U}$	2	0	-1

Thus, if (V, ρ_V) is the trivial representation of S_3, then

$$\begin{aligned}
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{\text{triv}} \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G V_{\text{triv}}} \rangle = \frac{1}{6}(1+3+2) = 1 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{\text{sign}} \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G V_{\text{sign}}} \rangle = \frac{1}{6}(1-3+2) = 0 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_W \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G W} \rangle = \frac{1}{6}(3+3+0) = 1 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{W'} \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G W'} \rangle = \frac{1}{6}(3-3+0) = 0 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_U \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G U} \rangle = \frac{1}{6}(2+0-2) = 0
\end{aligned}$$

It follows that $\operatorname{Ind}_H^G V \cong V_{\text{triv}} \oplus W$.

If (V, ρ_V) is the sign representation of S_3, then

$$\begin{aligned}
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{\text{triv}} \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G V_{\text{triv}}} \rangle = \frac{1}{6}(1-3+2) = 0 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{\text{sign}} \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G V_{\text{sign}}} \rangle = \frac{1}{6}(1+3+2) = 1 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_W \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G W} \rangle = \frac{1}{6}(3-3+0) = 0 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{W'} \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G W'} \rangle = \frac{1}{6}(3+3+0) = 1 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_U \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G U} \rangle = \frac{1}{6}(2+0-2) = 0
\end{aligned}$$

It follows that $\operatorname{Ind}_H^G V \cong V_{\text{sign}} \oplus W'$.

Finally, if (V, ρ_V) is the irreducible 2-dimensional representation of S_3, then $\chi_V(3) = 2$, $\chi_V(1\ 2) = 0$, and $\chi_V(1\ 2\ 3) = -1$. Then

$$\begin{aligned}
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{\text{triv}} \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G V_{\text{triv}}} \rangle = \frac{1}{6}(2+0-2) = 0 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{\text{sign}} \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G V_{\text{sign}}} \rangle = \frac{1}{6}(2+0-2) = 0 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_W \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G W} \rangle = \frac{1}{6}(6+0+0) = 1 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{W'} \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G W'} \rangle = \frac{1}{6}(6+0+0) = 1 \\
\langle \chi_{\operatorname{Ind}_H^G V}, \chi_U \rangle &= \langle \chi_V, \chi_{\operatorname{Res}_H^G U} \rangle = \frac{1}{6}(4+0+2) = 1
\end{aligned}$$

It follows that $\operatorname{Ind}_H^G V \cong W \oplus W' \oplus U$.

4. We first compute the inner product of $\chi_{\operatorname{Ind}_H^G V}$ with itself: $\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{\operatorname{Ind}_H^G V} \rangle = \sum_i d_i^2$. But Frobenius reciprocity tells us that this inner product is equal to $\langle \chi_V, \chi_{\operatorname{Res}_H^G \operatorname{Ind}_H^G V} \rangle$,

which is equal to the number of times V occurs in the decomposition of $\operatorname{Res}_H^G \operatorname{Ind}_H^G V$. But V is irreducible and $\dim \operatorname{Res}_H^G \operatorname{Ind}_H^G V = [G:H] \dim V$, so $\langle \chi_V, \chi_{\operatorname{Res}_H^G \operatorname{Ind}_H^G V} \rangle \leq [G:H]$.

5. It is enough to check that the characters of $\operatorname{Ind}_H^G V$ and $\operatorname{Ind}_H^G V_{g_i}$ agree. Let $W = V_{g_i}$ and let $\rho_W := \rho_{\operatorname{Ind}_H^G V}|_H$ be the representation of H on W. Then if $g_j g g_j^{-1} \in H$, we have seen that $\rho_W(g_j g g_j^{-1}) = \rho_V(g_i g_j g g_j^{-1} g_i^{-1})$ as linear transformations $W = V_{g_i} \to W = V_{g_i}$. Now

$$\chi_{\operatorname{Ind}_H^G W}(g) = \sum_{j: g_j g g_j^{-1} \in H} \chi_W(g_j g g_j^{-1}) = \sum_j \chi_W(g_j g g_j^{-1}) = \sum_j \chi_V(g_i g_j g g_j^{-1} g_i^{-1})$$

Note that we have used the assumption that $H \lhd G$ is a *normal* subgroup. But $\{g_i g_j\}_j = \{g_j\}_j$, so we may rearrange the last sum to get

$$\chi_{\operatorname{Ind}_H^G W}(g) = \sum_j \chi_V(g_i g_j g g_j^{-1} g_i^{-1}) = \sum_j \chi_V(g_j g g_j^{-1}) = \chi_{\operatorname{Ind}_H^G V}(g)$$

as desired.