M3/4/5P12 SOLUTIONS FOR PROGRESS TEST #1

Question 1.

(1) A subrepresentation is a vector subspace $W \subset V$ such that $\rho_V(g) : V \to V$ preserves W for each $g \in G$. Equivalently, $\rho_V(g)(w) \in W$ for all $w \in W$ and all $g \in G$.

(2) We first check that $V^G \subset V$ is a vector subspace. Indeed, if $v_1, v_2 \in V^G$, then for any $\lambda_1, \lambda_2 \in \mathbf{C}$,

$$\rho_V(g)(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \rho_V(g)(v_1) + \lambda_2 \rho_V(g)(v_2) = \lambda_1 v_1 + \lambda_2 v_2$$

so $\lambda_1 v_1 + \lambda_2 v_2 \in V^G$ and V^G is a vector space.

If $v \in V^G$ and $g \in G$, then $\rho_V(g)(v) = v \in V^G$, so $\rho_V(g)$ preserves V^G for all $g \in G$.

(3) Define $\pi : V \to V$ by setting

$$\pi(v) := \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v)$$

We first need to check that $\mathrm{im}(\pi) \subset V^G$: For any $g' \in G$,

$$\begin{aligned}
\rho_V(g')(\pi(v)) &= \frac{1}{|G|} \sum_{g \in G} \rho_V(g')\rho_V(g)(v) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_V(g'g)(v) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v) \\
&= \pi(v)
\end{aligned}$$

since $\{g'g\}_{g \in G} = G$. In addition, π is a linear transformation, since each $\rho_V(g) : V \to V$ is a linear transformation and π is a linear combination of the maps $\rho_V(g)$. Finally,

$$\begin{aligned}
\pi(\rho_V(g')(v)) &= \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(\rho_V(g')(v)) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_V(gg')(v) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho_V(g)(v) \\
&= \pi(v) \\
&= \rho_V(g')(\pi(v))
\end{aligned}$$

since $\{gg'\}_{g \in G} = G$, so π is G-linear.

(4) V^G is generated by the elements $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

To write down π as in part (3), it suffices to write down $\pi \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\pi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\pi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Since $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in V^G$,

$$\pi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \pi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

In addition,

$$\begin{aligned}
\pi \begin{pmatrix}1\\0\\0\end{pmatrix} &= \frac{1}{4}\sum_{g\in C_4}\rho_V(g)\begin{pmatrix}0\\1\\0\end{pmatrix}\\
&= \frac{1}{4}\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}i&0&0\\i-1&1&0\\i-1&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}-1&0&0\\-2&1&0\\-2&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}-i&0&0\\-i-1&1&0\\-i-1&0&1\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}\right)\\
&= \frac{1}{4}\left(\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}i\\i-1\\i-1\end{pmatrix} + \begin{pmatrix}-1\\-2\\-2\end{pmatrix} + \begin{pmatrix}-i\\-i-1\\-i-1\end{pmatrix}\right) = \begin{pmatrix}0\\-1\\-1\end{pmatrix}
\end{aligned}$$

Thus, the matrix for $\pi : V \to V^G$ is $\begin{pmatrix}0&0&0\\-1&1&0\\-1&0&1\end{pmatrix}$. Its kernel is generated by $\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right)$, and

$$\rho_V(g)\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}i&0&0\\i-1&1&0\\i-1&0&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix} = i\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Thus, $V \cong V_{\text{triv}} \oplus V_{\text{triv}} \oplus V_1$, where $(V_{\text{triv}}, \rho_{\text{triv}})$ is the 1-dimensional trivial representation and (V_1, ρ_1) is the 1-dimensional representation with $\rho_1(g) = i$.

Question 2.

(1) Suppose (V, ρ_V) is a 1-dimensional representation of S_4. Then for any transposition $(a\ b) \in S_4$, $\rho_V(a\ b)^2 = 1$, so $\rho_V(a\ b) = \pm 1$. Since every transposition is conjugate to $(1\ 2)$, $\rho_V(a\ b) = \rho_V(1\ 2)$ and there are only two possibilities: the trivial representation, where $\rho_V(g) = 1$ for all $g \in S_4$, and the sign representation, where $\rho_V(g)$ is 1 for even permutations and -1 for odd permutations (which includes all transpositions).

(2) The vector $\left(\begin{smallmatrix}1\\1\\1\\1\end{smallmatrix}\right)$ generates the trivial subrepresentation.

Suppose $\left(\begin{smallmatrix}c_1\\c_2\\c_3\\c_4\end{smallmatrix}\right)$ generates a subrepresentation. It must be isomorphic to either the trivial representation or the sign representation, so $\left(\begin{smallmatrix}c_1\\c_2\\c_3\\c_4\end{smallmatrix}\right)$ is a simultaneous eigenvector for the above six matrices, with eigenvalue ± 1. This eigenvalue 1 for each matrix if the subrepresentation is isomorphic to the trivial representation, and it must be -1 for each matrix if the subrepresentation is isomorphic to the sign representation.

But those matrices amount to swapping c_a and c_b for all pairs (a, b), so if the eigenvalue were -1, we would have $c_a = -c_b$ for all pairs (a, b), implying $c_1 = c_2 = c_3 = c_4 = 0$. Thus, the eigenvalue must be 1. We then see (by applying the matrix for the transpositions $(a\ b)$) that $c_1 = c_b$ for all pairs (a, b). Therefore, $c_1 = c_2 = c_3 = c_4$, so our eigenvector is a multiple of $\left(\begin{smallmatrix}1\\1\\1\\1\end{smallmatrix}\right)$. Thus, there are no other 1-dimensional subrepresentations of V.

(3) Let W be the 1-dimensional trivial representation above. Maschke's theorem tells us that there is a complementary subrepresentation $W' \subset V$, which must be 3-dimensional. If W' were reducible, it would have a 1-dimensional subrepresentation, but W is the only 1-dimensional subrepresentation of V.

(4) We have already found two 1-dimensional representations of S_4 and we have proved that there is a 3-dimensional irreducible representation of S_4. But

$$|S_4| = 24 = \sum_i (\dim W_i)^2$$

where the W_i run over the irreducible representations of S_4. Therefore, if we denote the irreducible representations we have already found by W_1, W_2, and W_3, we have

$$24 = 1^2 + 1^2 + 3^2 + \sum_{i>3} (\dim W_i)^2$$

so $13 = \sum_{i>3} (\dim W_i)^2$. Since $\dim W_i \geq 2$ for $i > 3$, the only way to write 13 as a sum of squares is $13 = 2^2 + 3^2$. Thus, the irreducible representations of S_4 have dimensions 1, 1, 2, 3,and 3.