M3/4/5P12 Problem sheet #5

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1. If $f : G \to H$ is a homomorphism of finite groups, prove that f induces an algebra homomorphism $\mathbf{C}[G] \to \mathbf{C}[H]$.
2. Write down an isomorphism between D_8 and a direct sum of matrix algebras.
3. Find two groups, G and H, which are not isomorphic but which have isomorphic group algebras, $\mathbf{C}[G]$ and $\mathbf{C}[H]$. Thus, not every homomorphism of group algebras comes from a homomorphism of groups.
4. Show directly that $\mathrm{Mat}_2(\mathbf{C})$ has no 1-dimensional modules.
5. Prove that the 3-dimensional algebra $A = \mathbf{C}[x]/x^3$, which has basis $\{1, x, x^2\}$ and multiplication given by $x \cdot x = x^2$ and $x \cdot x^2 = x^2 \cdot x = 0$, is not semisimple.
6. Prove that if A is an algebra and M is a finite-dimensional $\mathbf{C}[G]$-module, then $\mathrm{Hom}(M, \mathbf{C})$ is naturally an A^{op}-module. If $A = \mathbf{C}[G]$ for a finite group G and M corresponds to a representation of G, compare your construction with the dual representation.
7. Suppose M and N are A-modules and we attempt to make $M \otimes N$ into an A-module by setting $a \cdot (m \otimes n) = (a \cdot m) \otimes (a \cdot n)$. What goes wrong?
8. *(Advanced)* Find an algebra which is not isomorphic to its opposite algebra.