M3/4/5P12 Problem sheet #4

Rebecca Bellovin

1. Consider the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication given by $i^2 = j^2 = k^2 = ijk = -1$ (for this group, we use 1 to denote the identity element of the group).
   (a) Find the conjugacy classes of Q_8.
   (b) Prove that Q_8 and D_8 are not isomorphic.
   (c) Find all of the 1-dimensional representations of Q_8.
   (d) Write down the character table for Q_8.
2. Let G be a finite group such that every irreducible representation of G is 1-dimensional. Show that G is abelian. *Hint*: find the number of conjugacy classes in G.
3. Let $G = S_3$.
   (a) Write down the character table of G.
   (b) Let (V, ρ_V) be the 2-dimensional irreducible representation of G. Find the characters of V^*, $V \otimes V$, and $\mathrm{Hom}(V, V)$.
   (c) Write the characters of $V \otimes V$ and $\mathrm{Hom}(V, V)$ as linear combinations of irreducible characters.
   (d) Consider the class function $\phi : S_3 \to \mathbf{C}$ defined by $\phi(e) = 4$, $\phi(1\ 2) = 0$, and $\phi(1\ 2\ 3) = -5$. Write ϕ as a linear combination of irreducible characters of S_3. Is ϕ the character of a representation of S_3?
4. We will work out the character table of $A_4 \subset S_4$, which is the group of even permutations. There are four conjugacy classes, with representatives e, $(1\ 2\ 3)$, $(1\ 3\ 2)$, and $(1\ 2)(3\ 4)$, and sizes 1, 4, 4, and 3, respectively.
   (a) Prove that the class function $\chi_U : A_4 \to \mathbf{C}$ given by $\chi_U(e) = 3$, $\chi_U((1\ 2\ 3)) = \chi_U((1\ 3\ 2)) = 0$, and $\chi_U((1\ 2)(3\ 4)) = -1$ is an irreducible character. What is the dimension of the corresponding representation? *Hint*: Look at the character table for S_4.
   (b) Show that A_4 has two more irreducible representations, both 1-dimensional.
   The character table is now

	e	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(1\ 2)(3\ 4)$
size of conjugacy class	1	4	4	3
$\chi_1 = \chi_{\mathrm{triv}}$	1	1	1	1
χ_U	3	0	0	-1
χ_3	1	?	?	?
χ_4	1	?	?	?

   (c) Use row orthogonality relations to show that $\chi_3((1\ 2)(3\ 4)) = \chi_4((1\ 2)(3\ 4)) = 1$.
   (d) Fill in the rest of the character table.
   (e) (*Advanced*) Show that χ_3 and χ_4 are obtained by inflating 1-dimensional representations of C_3.
5. Let $G \subset S_4$ be the subgroup generated by $(1\ 2\ 3\ 4)$ and $(1\ 2)(3\ 4)$. Then $G \cong D_8$. Using the character tables for D_8 and S_4, if (V, ρ_V) is an irreducible representation of S_4, write down the irreducible decomposition of its restriction to G.

6. Find an algebra isomorphism between $\mathbf{C}[C_3]$ and $\mathbf{C} \oplus \mathbf{C} \oplus \mathbf{C}$. More generally, find an algebra isomorphism between $\mathbf{C}[C_n]$ and $\mathbf{C}^{\oplus n}$.

7. Let A and B be algebras. Show that the projection map $\pi : A \oplus B \to A$ is an algebra homomorphism. Show that the inclusion $\sigma : A \to A \oplus B$ is not an algebra homomorphism.