M3/4/5P12 Problem sheet #2

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1. Let (V, ρ_V) and (W, ρ_W) be finite-dimensional complex representations of a finite group G.
   (a) Define an action of G on $\mathrm{Hom}(V, W)$ (the set of linear transformations from V to W) by setting $g \cdot f := \rho_W(g) \circ f \circ \rho_V(g^{-1})$. Prove that this defines a representation of G and calculate the dimension of the representation.
   (b) Prove that $f : V \to W$ is G-linear if and only if $g \cdot f = f$.
2. Suppose we want to consider representations over fields other than $\mathbf{C}$. For example, consider the homomorphism $\rho : C_2 \to \mathrm{GL}_2(\mathbf{F}_2)$ given by $\rho(g) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Is this representation irreducible? Is it decomposable?
3. The action of $D_8 = \langle s, t : s^4 = t^2 = e, tst = s^{-1} \rangle$ on the square defines a 4-dimensional representation. Is this representation irreducible? If not, write down its irreducible subrepresentations (it is enough to specify the representations on s and t).
4. We have previously defined a 2-dimensional representation of D_8, given by $s \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $t \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Prove that this representation is irreducible. Are there any other irreducible 2-dimensional representations of D_8 (up to isomorphism)?
5. (a) Let G be a finite group, and let $\chi : G \to \mathrm{GL}_1(\mathbf{C})$ be a 1-dimensional matrix representation. Show that for all $g, h \in G$, $\chi(hgh^{-1}) = \chi(g)$.
   (b) Let $G = S_n$. Prove that all transpositions are conjugate to (12).
   (c) Prove that there are only two 1-dimensional representations of S_n (up to isomorphism).
6. Let (V, ρ_V) be an irreducible representation of a finite group G. Compute $\mathrm{Hom}(V^{\oplus r}, V^{\oplus r})^G$. *Hint*: Consider the case $r = 1$.
7. (a) Let $G = C_n$. Write down all of the 1-dimensional representations of G. Show that each one appears as a subrepresentation of the regular representation of G.
   (b) Let G be a finite abelian group. How many isomorphism classes of irreducible complex representations are there?
8. *Advanced question* Let $G = D_{2n} = \langle t, s : t^n = s^2 = e, sts = t^{-1} \rangle$. We will classify 2-dimensional representations of G.
   (a) Let (V, ρ_V) be a 2-dimensional complex representation of G. Show that there is a basis $\mathcal{B}_V = (b_1, b_2)$ such that $\rho_{V,\mathcal{B}_V}(t)$ is diagonal. What form does this matrix have?
   (b) What are the possibilities for $\rho_{V,\mathcal{B}_V}(s)$?
   (c) When is V reducible?
   (d) Show that if n is even, there are $(n-2)/2$ isomorphism classes of 2-dimensional representations of G.
   (e) Show that if n is odd, there are $(n-1)/2$ isomorphism classes of 2-dimensional representations of G.