M3/4/5P12: INDUCED REPRESENTATIONS

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1. Definitions and properties

1.1. **Definition.** Suppose G is a finite group and $H \subset G$ is a subgroup. Then if (V, ρ_V) is a representation of G, we may restrict $\rho_V : G \to \mathrm{GL}(V)$ to H to get a homomorphism $\rho_V|_H : H \to \mathrm{GL}(V)$. We denote this representation of H by $(\mathrm{Res}_H^G V, \rho_V|_H)$.

The theory of induced representations lets us go the other way: Given a representation (W, ρ_W) of H, $\mathrm{Ind}_H^G V$ will denote the vector space underlying a representation of G.

The constructions we give are different from the "standard" constructions; we have done this to avoid having to talk about tensor products of modules over non-commutative rings.

Definition 1.1. Let $H \subset G$ be a subgroup and let (V, ρ_V) be a representation of H. Then

$$\mathrm{Ind}_H^G V := \{f : G \to V : f(hg) = \rho_V(h)f(g) \text{ for all} h \in H, g \in G\}$$

We define an action $G \times \mathrm{Ind}_H^G V \to \mathrm{Ind}_H^G V$ by setting $(g \cdot f)(g') = f(g'g)$, and we write $\rho_{\mathrm{Ind}_H^G V}$ for the associated representation.

Thus, Ind_H^G is the set of functions $f : G \to V$ which are equivariant for multiplication by H.

Now we fix coset representatives $g_1, \ldots, g_s \in G$, so that $G = \coprod_i Hg_i$.

Lemma 1.2. *If $f \in \mathrm{Ind}_H^G V$, then f is determined by its values on $\{g_i\}_i$.*

Proof. By definition, $f(hg_i) = \rho_V(h)f(g_i)$. Since any element $g \in G$ can be written uniquely in the form $g = g_i h$, the set of values $\{f(g_i)\}_i$ determines f. $\square$

Example 1.3. Suppose $H = \{e\}$ and (V, ρ_V) is the trivial representation. Then $\mathrm{Ind}_{\{e\}}^G V = \{f : G \to V\}$, which has basis $\{\delta_g\}_{g \in G}$, where $\delta_g(h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{if } g \neq h \end{cases}$, and G acts via $h \cdot \delta_g = \delta_{gh^{-1}}$.

But we can define a map $V_{\mathrm{reg}} \to \mathrm{Ind}_{\{e\}}^G$ via $b_g \mapsto \delta_{g^{-1}}$. This is an isomorphism of vector spaces, and

$$(\rho_{\mathrm{reg}}(h)(b_g)) = b_{hg} \mapsto \delta_{g^{-1}h^{-1}} = h \cdot \delta_{g^{-1}}$$

so it is G-linear. Thus, $\mathrm{Ind}_{\{e\}}^G$ is naturally isomorphic to V_{reg}.

Proposition 1.4. *Let (V, ρ_V) and $(V', \rho_{V'})$ be representations of H and let $\varphi : V \to V'$ be an H-linear map. Then there is a G-linear map $\mathrm{Ind}_H^G \varphi : \mathrm{Ind}_H^G V \to \mathrm{Ind}_H^G V'$.*

Proof. We may define $\operatorname{Ind}_H^G \varphi$ by defining $(\operatorname{Ind}_H^G \varphi)(f)(g) := \varphi \circ f : G \to V'$ for $f \in \operatorname{Ind}_H^G V$. Then

$$(\operatorname{Ind}_H^G \varphi)(g' \cdot f)(g) = (\varphi \circ g' \cdot f)(g) = \varphi(f(gg')) = (g' \cdot (\varphi \circ f))(g)$$

so $\operatorname{Ind}_H^G \varphi$ is G-linear. $\square$

Definition 1.5. The *support* of a function $f : G \to V$ is the set $\{g \in G : f(g) \neq 0\}$.

If $f \in \operatorname{Ind}_H^G V$, then since $f(hg) = \rho_V(h)f(g)$ for all $h \in H$, $g \in G$, if $f(g) \neq 0$ then $f(hg) \neq 0$. Thus, the support of f is a union of right cosets.

If $g \in G$, we define $V_g \subset \operatorname{Ind}_H^G V$ to be the subspace of functions $f \in \operatorname{Ind}_H^G V$ whose support is contained in Hg.

Lemma 1.6. *There is an H-linear isomorphism*

$$\begin{aligned} ev : V_e &\to V \\ f &\mapsto f(e) \end{aligned}$$

Proof. Observe that $V_e = \{f : H \to V : f(h) = \rho_V(h)f(e) \text{ for } h \in H\}$. Thus, $f \in V_e$ is determined by $f(e)$ and any choice of $f(e) \in V$ determines an element of V_e. Thus, we have an isomorphism of vector spaces.

Furthermore,

$$\rho_{\operatorname{Ind}_H^G V}(h)(f)(e) = f(e \cdot h) = f(h) = \rho_V(h)f(e)$$

so the map is H-linear. $\square$

Lemma 1.7. *For any $g \in G$, $\rho_{\operatorname{Ind}_H^G V}(g) : \operatorname{Ind}_H^G V \to \operatorname{Ind}_H^G V$ carries V_e isomorphically to $V_{g^{-1}}$, with inverse $\rho_{\operatorname{Ind}_H^G V}(g^{-1})$. Consequently, $\rho_{\operatorname{Ind}_H^G V}(g)$ carries $V_{g'}$ isomorphically to $V_{g'g^{-1}}$.*

Proof. Let $f \in V_e$. Then

$$(\rho_{\operatorname{Ind}_H^G V}(g)(f))(g') = f(g'g)$$

which is 0 unless $g'g \in H$, or equivalently, unless $g' \in Hg^{-1}$. Thus, $(\rho_{\operatorname{Ind}_H^G V}(g)(f)) \in V_{g^{-1}}$.

On the other hand, if $f \in V_{g^{-1}}$, then

$$(\rho_{\operatorname{Ind}_H^G V}(g^{-1})(f))(g') = f(g'g^{-1})$$

which is 0 unless $g'g^{-1} \in Hg^{-1}$, or equivalently, unless $g' \in H$. Thus, $(\rho_{\operatorname{Ind}_H^G V}(g)(f)) \in V_e$. $\square$

Corollary 1.8. *Each subspace $V_g \subset \operatorname{Ind}_H^G V$ is isomorphic (as a vector space) to V.*

Corollary 1.9. *We fix a set of coset representatives $g_1, \ldots, g_s$ for $H\mathbf{G}$. Then the natural map*

$$\begin{aligned} \oplus_i V_{g_i} &\to \operatorname{Ind}_H^G V \\ (f_i) &\mapsto \sum_i f_i \end{aligned}$$

is an isomorphism. It follows that $\dim \operatorname{Ind}_H^G V = [G : H] \dim V$.

Proof. If $f \in \operatorname{Ind}_H^G V$, we define $f_i(g) = \begin{cases} f(g) & \text{if } g \in Hg_i \\ 0 & \text{if } g \notin Hg_i \end{cases}$. Then $f \mapsto (f_i)_i$ is an inverse to the map above. $\square$

Example 1.10. Suppose (V, ρ_V) is the regular representation of H. Then we define a G-linear map $V_{\text{reg}} \to \operatorname{Ind}_H^G V$, where $(V_{\text{reg}}, \rho_{\text{reg}})$ is the regular representation of G, by sending

$$b_g \mapsto \left(g' \mapsto \begin{cases} b_h & \text{if } g' = hg^{-1} \text{ for } h \in H \\ 0 & \text{if } g' \notin Hg^{-1} \end{cases} \right)$$

In other words, we choose a function $f : G \to V$ which is 0 outside Hg^{-1} and sends hg^{-1} to b_h.

Example 1.11. For a more interesting example, consider $G = D_{2n} = \langle s, t : s^n = t^2 = e, tst = s^{-1} \rangle$ and $H = C_n = \langle s : s^n = e \rangle \subset G$. Then a 1-dimensional representation (V, ρ_V) of H is given by $g \mapsto \zeta$, where $\zeta^n = 1$. We compute $\operatorname{Ind}_H^G V$.

Corollary 1.9 implies that $\operatorname{Ind}_H^G V$ is 2-dimensional, and as a vector space $\operatorname{Ind}_H^G V \cong V_e \oplus V_t$, where V_e and V_t are 1-dimensional. Furthermore, Lemma 1.7 implies that $\rho_{\operatorname{Ind}_H^G V}(s)$ preserves V_e and carries V_t isomorphicallly to $V_{ts} = V_t$. We actually have a commutative diagram

$$\begin{tikzcd}
V_t \arrow[r, "\rho_{\operatorname{Ind}_H^G V}(s)"] & V_t \arrow[d, "\rho_{\operatorname{Ind}_H^G V}(t)"] \\
V_e \arrow[u, "\rho_{\operatorname{Ind}_H^G V}(t)"] \arrow[r, "\rho_{\operatorname{Ind}_H^G V}(tst)"] & V_e
\end{tikzcd}$$

so $\rho_{\operatorname{Ind}_H^G V}(s) : V_t \to V_t$ acts via $\rho_V(s^{-1})$.

Thus, if we choose a basis vector $v \in V$, we get a basis vector $v' \in V_e$ and we may choose a basis vector $\rho_{\operatorname{Ind}_H^G V}(t)(v') \in V_t$. With respect to this basis, the matrix representation for $\operatorname{Ind}_H^G V$ is

$$\rho_{\operatorname{Ind}_H^G V}(s) = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \qquad \text{and} \qquad \rho_{\operatorname{Ind}_H^G V}(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

1.2. **Induction of $\mathbf{C}[H]$-modules.** We can rephrase induced representations in the language of modules over group algebras. Recall that if $H \subset G$ is a subgroup, then $\mathbf{C}[H] \subset \mathbf{C}[G]$. Then multiplication makes $\mathbf{C}[G]$ a $\mathbf{C}[H]$-module.

Definition 1.12. Let M be a $\mathbf{C}[H]$]-module, and consider the space $\operatorname{Hom}_{\mathbf{C}[H]}(\mathbf{C}[G], M)$. We define an action of $\mathbf{C}[G]$ on $\operatorname{Hom}_{\mathbf{C}[H]}(\mathbf{C}[G], M)$ by setting $(a \cdot f)(b) = f(ba)$.

We check that with this $\mathbf{C}[G]$-module structure, $\operatorname{Hom}_{\mathbf{C}[H]}(\mathbf{C}[G], M)$ corresponds to the representation of G on $\operatorname{Ind}_H^G M$:

Lemma 1.13. *The map*

$$\operatorname{Ind}_H^G M \to \operatorname{Hom}_{\mathbf{C}[H]}(\mathbf{C}[G], M)$$
$$f \mapsto (\sum_{g \in G} \lambda_g [g] \mapsto \sum_{g \in G} \lambda_g f(g)$$

is an isomorphism of $\mathbf{C}[G]$*-modules.*

Proof. To check that this map is $\mathbf{C}[G]$-linear, it suffices to check that

$$[h] \cdot f \mapsto \left(\sum_{g \in G} \lambda_g [g] \mapsto \sum_{g \in G} \lambda_g f(gh) \right)$$

But $([h] \cdot f)(g) = f(gh)$, so

$$[h] \cdot f \mapsto \left(\sum_{g \in G} \lambda_g [g] \mapsto \sum_{g \in G} \lambda_g ([h] \cdot f)(g) \right) = \left(\sum_{g \in G} \lambda_g [g] \mapsto \sum_{g \in G} \lambda_g f(gh) \right)$$

as desired.

We write down the inverse map

$$\begin{aligned} \operatorname{Hom}_{\mathbf{C}[H]}(\mathbf{C}[G], M) &\to \operatorname{Ind}_H^G M \\ f &\mapsto (g \mapsto f([g])) \end{aligned}$$

so we have an isomorphism. □

Remark 1.14. If you know about tensor products of modules over non-commutative algebras, here we are viewing M as a left $\mathbf{C}[H]$-module and $\mathbf{C}[G]^*$ as a right $\mathbf{C}[H]$-module, and we are taking the tensor product over $\mathbf{C}[H]$.

2. Characters

If (V, ρ_V) is a representation of H, we wish to work out the character of the induced representation $(\operatorname{Ind}_H^G V, \rho_{\operatorname{Ind}_H^G V})$.

Proposition 2.1. *Let $g_1, \ldots, g_s$ be representatives for the cosets of H. Then $\chi_{\operatorname{Ind}_H^G V}(g) = \sum_{i : g_i g g_i^{-1} \in H} \chi_V(g_i g g_i^{-1})$.*

Proof. Corollary 1.9 shows that as a vector space, $\operatorname{Ind}_H^G V \cong \oplus_i V_{g_i}$, and Lemma 1.7 shows that $\rho_{\operatorname{Ind}_H^G V}(g)$ carries V_{g_i} isomorphically to $V_{g_i g^{-1}}$ for any $g \in G$. To compute the trace of $\rho_{\operatorname{Ind}_H^G V}(g)$, we only need to consider the g_i such that $Hg_i = Hg_i g^{-1}$, i.e., such that $g_i g g_i^{-1} \in H$.

Suppose $g_i g g_i^{-1} \in H$. Then we use the fact that V_{g_i} is isomorphic to V via $\rho_{\operatorname{Ind}_H^G V}(g_i^{-1}) : V_e \xrightarrow{\sim} V_{g_i}$. That is, we have a commutative square

$$\begin{tikzcd} V_{g_i} \arrow[r, "\rho_{\operatorname{Ind}_H^G V}(g)"] & V_{g_i} \arrow[d, "\rho_{\operatorname{Ind}_H^G V}(g_i)"] \\ V_e \arrow[u, "\rho_{\operatorname{Ind}_H^G V}(g_i^{-1})"] \arrow[r, "\rho_{\operatorname{Ind}_H^G V}(g_i g g_i^{-1})"] & V_e \end{tikzcd}$$

Thus, the trace of $\rho_{\operatorname{Ind}_H^G V}(g)$ on V_{g_i} is equal to the trace of $\rho_{\operatorname{Ind}_H^G V}(g_i g g_i^{-1})$ on V_e. But since V_e is isomorphic to V as a representation of H, this trace is $\chi_V(g_i g g_i^{-1})$.

Thus, we see that

$$\operatorname{Tr} \rho_{\operatorname{Ind}_H^G V}(g) = \sum_{i : g_i g g_i^{-1} \in H} \chi_V(g_i g g_i^{-1})$$

□

We can rewrite this formula slightly so as not to depend on our choice of coset representatives:

Corollary 2.2. $\chi_{\mathrm{Ind}_H^G V}(g) = \frac{1}{|H|}\sum_{g' \in G : g' g g'^{-1} \in H} \chi_V(g' g g'^{-1})$

Proof. If $g_i g g_i^{-1} \in H$, then so is $(hg_i)g(hg_i)^{-1}$ for $h \in H$. Thus, if we want the sum to run over all elements of G, instead of the coset representatives, we need to divide by $|H|$. □

3. Frobenius reciprocity

We return to the setting of representations. Lemma 1.6 implies that if (V, ρ_V) is a representation of H, then $\mathrm{Res}_H^G \mathrm{Ind}_H^G V$ contains a subspace we denoted V_e, which is preserved by $\rho_{\mathrm{Ind}_H^G V}(h)$ for $h \in H$. Thus, V_e is a representation of H, and it is isomorphic to V as a representation of H. Thus, there are H-linear maps $V \to \mathrm{Ind}_H^G V$ and $\mathrm{Ind}_H^G V \to V$.

We can formulate this more generally:

Theorem 3.1 (Frobenius reciprocity)**.** *Let V be a representation of H and let W be a representation of G. There is an isomorphism*

$$\mathrm{Hom}(W, \mathrm{Ind}_H^G V)^G \cong \mathrm{Hom}(\mathrm{Res}_H^G W, V)$$

Proof. Lemma 1.6 shows that there is a map $\mathrm{Ind}_H^G V \to V$ given by $f \mapsto f(e)$, and since $h \cdot f \mapsto (h \cdot f)(e) = f(h) = \rho_V(h) f(e)$ for $h \in H$, this map is H-linear. Thus, if $\varphi \in \mathrm{Hom}(W, \mathrm{Ind}_H^G V)^G$, we see that

$$w \mapsto (\varphi(w))(e) \in V$$

is an H-linear map, and so an element of $\mathrm{Hom}(\mathrm{Res}_H^G W, V)^H$.

On the other hand, given $\psi \in \mathrm{Hom}(\mathrm{Res}_H^G W, V)^H$, we define a map $W \to \mathrm{Ind}_H^G V$ via

$$w \mapsto (g \mapsto \psi(\rho_W(g)(w)))$$

(that is, we set $e \mapsto \psi(w)$ and rigged the rest so that the map was G-linear) □

Remark 3.2. The isomorphism of Frobenius reciprocity is usually stated as

$$\mathrm{Hom}(\mathrm{Ind}_H^G V, W)^G \cong \mathrm{Hom}(V, \mathrm{Res}_H^G W)$$

The version we have stated holds because we are working with representations of finite groups.

Corollary 3.3. *There are natural maps $W \to \mathrm{Ind}_H^G \mathrm{Res}_H^G W$ and $\mathrm{Res}_H^G \mathrm{Ind}_H^G V \to V$; the first is G-linear and the second is H-linear.*

Proof. Take $V := \mathrm{Res}_H^G W$; the identity map $\mathrm{Res}_H^G W \to \mathrm{Res}_H^G W$ corresponds to a G-linear map $W \to \mathrm{Ind}_H^G \mathrm{Res}_H^G W$.

Similarly, we may take $W := \mathrm{Ind}_H^G V$; the identity map $\mathrm{Ind}_H^G V \to \mathrm{Ind}_H^G V$ corresponds to an H-linear map $\mathrm{Res}_H^G \mathrm{Ind}_H^G V \to V$. □

Corollary 3.4. *Let (V, ρ_V) be a representation of H and let (W, ρ_W) be a representation of H. Then*

$$\langle \chi_{\mathrm{Ind}_H^G V}, \chi_W \rangle = \langle \chi_V, \chi_{\mathrm{Res}_H^G W} \rangle$$

3.1. **Restrictions of induced representations.** We wish to understand $\operatorname{Res}_H^G \operatorname{Ind}_H^G V$ as a representation of H. For example, we know that there is an H-linear map $\operatorname{Res}_H^G \operatorname{Ind}_H^G V \to V$ (and in fact, $V_e \subset \operatorname{Res}_H^G \operatorname{Ind}_H^G V$ is a subrepresentation isomorphic to V).

We consider the case where $H \lhd G$ is a *normal* subgroup of G. If H is not normal in G, Mackey theory still gives an idea what happens, but it is more involved.

We again choose coset representatives $g_1, \ldots, g_s$ for $H \backslash G$ and we set $V_{g_i} \subset \operatorname{Ind}_H^G V$ to be the subspace of $f \in \operatorname{Ind}_H^G V$ with support in Hg_i. Then as a vector space, $\operatorname{Ind}_H^G V \cong \oplus_i V_{g_i}$.

We know that $\operatorname{Res}_H^G \operatorname{Ind}_H^G V$ is never irreducible (unless $H = G$), because it has dimension $[G : H] \dim V$ and contains a copy of V. We can say more:

Proposition 3.5. *If $H \lhd G$ is a normal subgroup, then $\rho_{\operatorname{Ind}_H^G V}(h)$ preserves V_{g_i} for all $h \in H$ and each g_i. Moreover, V_{g_i} is isomorphic (as a vector space) to $V \cong V_e$, and the action of $\rho_{\operatorname{Ind}_H^G V}(h)$ on V_{g_i} is given by $\rho_V(g_i h g_i^{-1})$.*

Proof. Lemma 1.7 implies that $\rho_{\operatorname{Ind}_H^G V}(h)$ carries V_{g_i} isomorphically to $V_{g_i h^{-1}}$, and since $g_i h g_i^{-1} \in H$ (since H is normal in G), $V_{g_i h^{-1}} = V_{g_i}$. The same lemma shows that we have a commutative diagram

$$\begin{tikzcd}
V_{g_i} \arrow[r, "\rho_{\operatorname{Ind}_H^G V}(h)"] & V_{g_i} \arrow[d, "\rho_{\operatorname{Ind}_H^G V}(g_i)"] \\
V_e \arrow[u, "\rho_{\operatorname{Ind}_H^G V}(g_i^{-1})"] \arrow[r, "\rho_{\operatorname{Ind}_H^G V}(g_i h g_i^{-1})"] & V_e
\end{tikzcd}$$

which gives the action of $\rho_{\operatorname{Ind}_H^G V}(h)$ on V_{g_i}. $\square$

Observe that the map $h \mapsto g_i h g_i^{-1}$ is an automorphism of H. However, $\rho_V(g_i h g_i^{-1})$ need not be equivalent to $\rho_V(h)$ as a matrix representation:

Example 3.6. Let $G = D_8 = \langle s, t : s^4 = t^2 = e, tst = s^{-1} \rangle$ and let $H = C_4 = \langle s : s^4 = e \rangle \subset G$. Let (V, ρ_V) be a 1-dimensional representation of H, given by $g \mapsto i^k$ for some $k \in \{0, 1, 2, 3\}$. Let e, t be our coset representatives. Then $\operatorname{Res}_H^G \operatorname{Ind}_H^G V \cong V_e \oplus V_t$, and s acts on V_t via multiplication by $\rho_V(tst) = \rho_V(s^{-1}) = i^{-k}$. This is equal to $\rho_V(s)$ if and only if $k \in \{0, 2\}$. Otherwise, the matrix representation of H on $\operatorname{Res}_H^G \operatorname{Ind}_H^G V$ is equivalent to $\left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$.

It is not difficult to check that the 1-dimensional representations (V, ρ_V) of $H = C_4$ with $\rho_V(s) = \pm i$ are exactly those such that $\operatorname{Ind}_H^G V$ is irreducible. In fact, this is a general phenomenon:

Proposition 3.7. *Suppose that (V, ρ_V) is an irreducible representation of H. Then the induced representation $\operatorname{Ind}_H^G V$ is irreducible if and only if none of the representations $(V_{g_i}, \rho_V(g_i h g_i^{-1}))$ for $Hg_i \neq H$ are isomorphic to (V, ρ_V).*

Proof. Recall that $\operatorname{Ind}_H^G V$ is irreducible if and only if $\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{\operatorname{Ind}_H^G V} \rangle = 1$. To compute this inner product, we use Frobenius reciprocity:

$$\begin{aligned}\langle \chi_{\operatorname{Ind}_H^G V}, \chi_{\operatorname{Ind}_H^G V} \rangle &= \langle \chi_{\operatorname{Res}_H^G \operatorname{Ind}_H^G V}, \chi_V \rangle \\ &= \sum_i \langle \chi_{V_{g_i}}, \chi \rangle\end{aligned}$$

But $\langle \chi_{V_{g_i}}, \chi \rangle$ is 1 if $V_{g_i} \cong V$ as a representation of H and 0 otherwise. $\square$

Corollary 3.8. *If (V, ρ_V) is the trivial representation of H, $\operatorname{Ind}_H^G V$ is reducible. In fact, it is $[G : H]$ copies of the trivial representation of G.*

Proof. Lemma 1.6 says that $V_e \cong V$ as a representation of H, and Proposition 3.5 says that the action of $\rho_{\operatorname{Ind}_H^G V}(h)$ on V_{g_i} is given by $\rho_V(g_i h g_i^{-1})$. But since $\rho_V : H \to \mathrm{GL}(V)$ is trivial, $\rho_{\operatorname{Ind}_H^G V}(h)$ acts as the identity on V_{g_i} for all i. $\square$

Example 3.9. Recall that $A_4 \subset S_4$ has three 1-dimensional representations (V_k, ρ_k), given by $(1\ 2\ 3) \mapsto \omega^k$, where $\omega = e^{2\pi i/3}$ and $k \in \{0, 1, 2\}$. Let e and $(1\ 2)$ be representatives for the cosets of $A_4 \backslash S_4$, and consider $\operatorname{Ind}_{A_4}^{S_4} V_i$.

Then $\operatorname{Ind}_{A_4}^{S_4} V_i \cong V_k \oplus V_{k,(1\ 2)}$ as a representation of A_4, and A_4 acts on $V_{k,(1\ 2)}$ via

$$(1\ 2\ 3) \mapsto \rho_k((1\ 2)(1\ 2\ 3)(1\ 2)) = \rho_k(1\ 3\ 2) = \rho_k(1\ 2\ 3)^2$$

Thus, $V_{k,(1\ 2)} \cong V_k$ (as a representation of A_4) if and only if $\rho_k(1\ 2\ 3) = \rho_k(1\ 2\ 3)^2$, which is the case if and only if $\rho_k(1\ 2\ 3) = 1$.

It follows that if $k \neq 0$, then $\operatorname{Ind}_{A_4}^{S_4} V_k$ is the unique irreducible 2-dimensional representation of S_4.