EVEN AND ODD PERMUTATIONS

This is a short note to describe explicitly which elements of S_4 (the symmetric group on 4 elements) and D_4 (the symmetries of a square) are even and which are odd. Recall that we can write every permutation σ as a product of transpositions; then σ is *even* if it is the product of an even number of transpositions, and *odd* if it is the product of an odd number of transpositions.

1. D_4

Let's number the vertices of our square as 1, 2, 3, and 4 (in order as we go around the square clockwise). In class we described the symmetries explicitly: There are 4 rotations of the square, which we can write as permutations

$$e, \qquad (1\,2\,3\,4), \qquad (1\,3)(2\,4), \qquad (1\,4\,3\,4)$$

There are 2 flips across the horizontal and vertical axes:

$$(1\,2)(3\,4), \qquad (1\,4)(2\,3)$$

And there are 2 flips across the diagonal axes:

$$(1\,3), \qquad (2\,4)$$

Thus, there are 4 even permutations, namely

$$e, \qquad (1\,3)(2\,4), \qquad (1\,2)(3\,4), \qquad (1\,4)(2\,3)$$

and there are 4 odd permutations, namely

$$(1\,2\,3\,4), \qquad (1\,4\,3\,4), \qquad (1\,3), \qquad (2\,4)$$

Recall that a k-cycle is the product of $k-1$ transpositions, so these length 4-cycles are the products of 3 transpositions, so they are odd.

We see that half of the elements of D_4 are even and half are odd.

2. S_4

The symmetric group S_4 has $4! = 24$ permutations, and we saw in class that half of them are even and half are odd. So we are looking for 12 even permutations (the elements of A_4) and 12 odd permutations.

As we saw in lecture, the identity element e is an even permutation.

Now we count transpositions (which will be odd): The data of a transposition consists exactly of a choice of 2 elements from $\{1, 2, 3, 4\}$, namely, which two elements are being swapped. So there are $\binom{4}{2} = 6$ transpositions, namely

$$(1\,2), \quad (1\,3), \quad (1\,4), \quad (2\,3), \quad (2\,4), \quad (3\,4)$$

Now we count products of 2 distinct transpositions (which will be even permutations). There are 2 kinds of such products: the transpositions can be disjoint, or they can have an element in common. In the first case, one transposition will swap the element 1 with something, and one will not. Moreover, once we've chosen our first transposition, the second simply swaps the 2 elements remaining. So we have 3 such products:

$$(1\,2)(3\,4), \quad (1\,3)(2\,4), \quad (1\,4)(2\,3)$$

In the second case, we observe that the product $(a\,b)(a\,c)$ is the 3-cycle $(a\,c\,b)$, so it's enough to count permutations represented by 3-cycles. If a 3-cycle contains 1, it has the form $(1\,a\,b)$, and there are 3 choices for a and then 2 choices for b. So this gives us 6 permutations:

$$(1\,2\,3), \quad (1\,2\,4), \quad (1\,3\,2), \quad (1\,3\,4), \quad (1\,4\,2), \quad (1\,4\,3)$$

If a 3-cycle in S_4 does not contain 1, it contains all of $2, 3, 4$, and there are 2 permutations we can make from them:

$$(2\,3\,4), \quad (2\,4\,3)$$

Check for yourself that all of these 3-cycles represent different permutations! But note that (for example) $(1\,2\,3)$ and $(2\,3\,1)$ represent the same permutation, so these really are all of the permutations in S_4 represented by 3-cycles.

There are 6 elements of S_4 left to find, and we will see they are all 4-cycles; since 4-cycles are products of 3 transpositions, they are odd. Now we can write any length-4 cycle in the form $(1\,a\,b\,c)$, where $\{a, b, c\} = \{2, 3, 4\}$. Thus, writing down a 4-cycle amounts to choosing an ordering for the elements of $\{2, 3, 4\}$, and there are 6 orderings. Explicitly, we have

$$(1\,2\,3\,4), \quad (1\,2\,4\,3), \quad (1\,3\,2\,4), \quad (1\,3\,4\,2), \quad (1\,4\,2\,3), \quad (1\,4\,3\,2)$$

Again, check for yourself explicitly that these are all different permutations.

We have found 24 elements of S_4, so we have listed every element. Counting up the even and odd permutations above, we see that we have 12 even permutations and 12 odd permutations, which is exactly what our theorem from class said should happen.